Math 116 — Second Midterm — March 25, 2024

EXAM SOLUTIONS

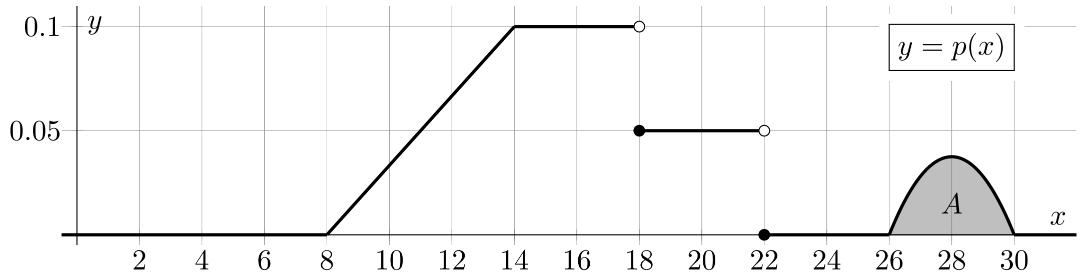
1. This exam has 15 pages including this cover.
2. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ notecard. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	9	
2	12	
3	13	
4	11	
5	14	

Problem	Points	Score
6	10	
7	7	
8	6	
9	12	
10	6	
Total	100	

1. [9 points] Every Saturday during the summer, Dominic rides his bicycle in a national park. The distance he travels on his bicycle each Saturday varies.
Let $p(x)$ be the **probability density function** (pdf) for x, the distance (in miles) that Dominic bicycles on a Saturday. The graph of $p(x)$, shown below, has the following properties:

- $p(x)$ is piecewise linear for $x \leq 26$.
- $p(x)$ is nonzero only for $8 < x < 22$ and $26 < x < 30$.
- The area of the shaded region is A.



For each part of this problem, your answer should not involve the letter A. You do not need to show your work in this problem, but partial credit may be awarded for work shown clearly.

a. [1 point] Find the **minimum** distance that Dominic bicycles on a Saturday.

Solution: Since $p(x) = 0$ for $x < 8$, there is a 0% chance that Dominic bicycles less than 8 miles. Since $p(x) > 0$ for x slightly larger than 8, there is a nonzero chance that Dominic bicycles slightly more than 8 miles. Therefore the minimum distance he travels is 8 miles.

Answer: 8 miles

b. [2 points] Find the **median** distance that Dominic bicycles on a Saturday.

Solution: We must find the value of x for which $\int_{-\infty}^{x} p(t)\,dt = 0.5$. This is given by $x = 16$ because, by adding areas of triangles and rectangles,

$$\int_{-\infty}^{16} p(t)\,dt = \int_{8}^{14} p(t)\,dt + \int_{14}^{16} p(t)\,dt = \frac{1}{2}(6)(0.1) + 2(0.1) = 0.3 + 0.2 = 0.5.$$

Answer: 16 miles

c. [2 points] Use the fact that $p(x)$ is a probability density function to find the value of A.

Solution: Since $p(x)$ is a probability density function, we must have $\int_{-\infty}^{\infty} p(t)\,dt = 1$. Adding up areas, we have

$$\int_{8}^{22} p(t)\,dt = \frac{1}{2}(6)(0.1) + 4(0.1) + 4(0.05) = 0.3 + 0.4 + 0.2 = 0.9.$$

Then $0.9 + A = 1$ implies $A = 0.1$.

Answer: $A =$ 0.1

d. [2 points] Calculate the probability that Dominic bicycles farther than 18 miles on a Saturday.

Solution: There are two possible ways to solve this problem:

Solution 1 (Using A)**:** Computing the probability directly, this is given by

$$\int_{18}^{\infty} p(t)\,dt = \int_{18}^{22} p(t)\,dt + A = 4(0.05) + A = 0.2 + A.$$

In part (c), we found that $A = 0.1$, so our answer is $0.2 + 0.1 = 0.3$.

Solution 2 (Without A)**:** The probability that Dominic bicycles for 18 miles or <u>less</u> is given by

$$\int_{-\infty}^{18} p(t)\,dt = \frac{1}{2}(6)(0.1) + 4(0.1) = 0.3 + 0.4 = 0.7.$$

Then the probability that Dominic bicycles for <u>more</u> than 18 miles is given by $1 - 0.7 = 0.3$.

Answer: 0.3 **or** 30%

e. [2 points] Complete the sentence below to write a practical interpretation of the equation $p(28) = 0.0375$:

The probability that Dominic bicycles between 27 and 29 miles on a Saturday is...

Solution: ...approximately $2(0.0375) = 0.075$, or 7.5%.

2. [12 points] Joe and Paula are at the same national park, hiking through the forest. They arrive at the bottom of a cliff and challenge each other to bring their hiking gear to the top of the cliff, which is 25 meters above the bottom. Each of them has a different idea of how to accomplish this.
You may assume that the acceleration due to gravity is $g = 9.8$ m/s^2.

a. [6 points] Joe plans to climb to the top of the cliff while carrying his water bottle. Before Joe starts climbing, the combined mass of Joe and his water bottle is 64 kilograms. However, a rock punctures the bottle as soon as Joe starts climbing, so water leaks out at a constant rate of 0.03 kilograms per second. Joe climbs the cliff at a constant rate of 0.25 meters per second.

(i) Let $M(h)$ be the combined mass of Joe and his water bottle, in kilograms (kg), when Joe is h meters above the ground. Write an expression for $M(h)$.

Solution: Water leaks out at a rate of 0.03 kg/s, and Joe climbs the cliff at a rate of 0.25 m/s, so $0.03/0.25 = 0.12$ kg of water leak out for every meter that Joe climbs. So the combined mass of Joe and the water bottle, when Joe is h meters above the ground, is $64 - 0.12h$ kg.

Answer: $M(h) =$ $64 - 0.12h$

(ii) Write an integral representing the total amount of work, in Joules (J), that it takes for Joe to move himself and the water bottle to the top of the cliff. Your answer should not involve the letter M. **Do not evaluate your integral.**

Solution: Below are two possible ways to solve this problem:

Solution 1 (Work in terms of height)**:** Recall that (work) = (force)(distance). The force equals $9.8(64 - 0.12h)$ N, as this is the mass $M(h) = 64 - 0.12h$ kg from part (i) times the acceleration 9.8 m/s^2. When Joe is at a height of h meters, the amount of work it takes for him to carry the water bottle an additional Δh meters is approximately $9.8(64 - 0.12h)\,\Delta h$ J. Therefore the total amount of work it takes for him to carry the water bottle 25 meters is

$$\int_0^{25} 9.8(64 - 0.12h)\,dh \text{ J.}$$

Solution 2 (Work in terms of time)**:** Alternatively, we can compute the work in terms of time without using our answer from part (i). The combined mass of Joe and the water bottle is 64 kilograms when Joe starts climbing the cliff, and this mass decreases at a rate of 0.03 kg/s due to the water bottle leaking. So, the combined mass of Joe and the water bottle t seconds after Joe starts climbing is $64 - 0.03t$ kg.

Recall that (work) = (force)(distance). The force equals $9.8(64 - 0.03t)$ N, as this is the mass $64 - 0.03t$ kg times the acceleration 9.8 m/s^2. To find the distance, we note that Joe climbs the cliff at a rate of 0.25 m/s. So, when he has been climbing the cliff for t seconds, he will climb an additional $0.25\,\Delta t$ m over the next Δt seconds. Thus the amount of work it takes for him to carry the water bottle an additional Δt seconds is approximately $9.8(0.25)(64 - 0.03t)\,\Delta t$ J. Since the cliff is 25 m tall, and Joe climbs the cliff at a rate of 0.25 m/s, then it takes $25/0.25 = 100$ seconds for Joe to climb the cliff. Therefore the total amount of work it takes for him to carry the water bottle 25 meters is

$$\int_0^{100} 9.8(0.25)(64 - 0.03t)\,dt \text{ J.}$$

Answer: $\int_0^{25} 9.8(64 - 0.12h)\,dh$ **OR** $\int_0^{100} 9.8(0.25)(64 - 0.03t)\,dt$

b. [6 points] Paula ties a rope to her 3-kilogram backpack, walks to the top of the cliff, and then uses the rope to pull her backpack to the top. The rope has a mass of 0.1 kilograms per meter. Write an integral representing the total amount of work, in Joules (J), that it takes for Paula to pull her backpack and the attached rope to the top of the cliff.
Do not evaluate your integral.

Solution: Below are two possible ways to solve this problem:

Solution 1 (Slicing the height)**:** When Paula has already lifted her backpack a total of h meters, the remaining rope she must pull up has a length of $25-h$ meters, and thus a mass of $0.1(25-h)$ kilograms. Together with the backpack, the amount of mass that remains for her to lift up is $3+0.1(25-h)$ kg. At this point, the amount of work it takes to lift the rope and the backpack an additional Δh meters is approximately $9.8(3+0.1(25-h))\Delta h$ J. Therefore the total amount of work it takes for her to pull the backpack up 25 meters is

$$\int_0^{25} 9.8(3+0.1(25-h))\,dh \text{ J}.$$

Solution 2 (Slicing the rope)**:** The work required to lift only the backpack up to the top of the cliff is $9.8(3)(25) = 9.8(75)$ J. Consider a segment of rope of length $\Delta\ell$ meters which is ℓ meters from the top of the cliff. The mass of this segment is $0.1\,\Delta\ell$, and the amount of work required to lift this segment to the top of the cliff is $9.8(0.1)\ell\,\Delta\ell$. Thus the amount of work required to lift the entire rope to the top of the cliff is $\int_0^{25} 9.8(0.1)\ell\,d\ell$. Therefore the total amount of work required to lift the backpack and the rope to the top of the cliff is

$$9.8(75) + \int_0^{25} 9.8(0.1)\ell\,d\ell \text{ J}.$$

Answer: $\int_0^{25} 9.8(3+0.1(25-h))\,dh$ **OR** $9.8(75) + \int_0^{25} 9.8(0.1)\ell\,d\ell$

3. [13 points] Consider the following sequences, each defined for $n \geq 1$:

$$a_n = \frac{\cos(\pi n)}{n} \qquad b_n = -\left(\frac{100}{99}\right)^n \qquad c_n = \sum_{k=0}^{n} \frac{1}{3^k}$$

a. [9 points] For each of the sequences above, determine whether the sequence is bounded, whether it is monotone, and whether it is convergent. No justification is required.

(i) The sequence a_n is...

Circle one: $\boxed{\textbf{Bounded}}$ **Unbounded**

Circle one: **Monotone** $\boxed{\textbf{Not Monotone}}$

Circle one: $\boxed{\textbf{Convergent}}$ **Divergent**

(ii) The sequence b_n is...

Circle one: **Bounded** $\boxed{\textbf{Unbounded}}$

Circle one: $\boxed{\textbf{Monotone}}$ **Not Monotone**

Circle one: **Convergent** $\boxed{\textbf{Divergent}}$

(iii) The sequence c_n is...

Circle one: $\boxed{\textbf{Bounded}}$ **Unbounded**

Circle one: $\boxed{\textbf{Monotone}}$ **Not Monotone**

Circle one: $\boxed{\textbf{Convergent}}$ **Divergent**

b. [4 points] Determine whether the following series is convergent or divergent. **Fully justify** your answer including using **proper notation** and **showing mechanics** of any tests you use. Circle your final answer choice.

$$\sum_{n=0}^{\infty} c_n$$

Circle one: **Convergent** $\boxed{\textbf{Divergent}}$

Solution: Note that

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{3^k} = \sum_{k=0}^{\infty} \frac{1}{3^k} = \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k = \frac{1}{1-\frac{1}{3}} = \frac{3}{2} \neq 0.$$

Therefore, by the ***n*th term test for divergence,** the series $\sum_{n=0}^{\infty} c_n$ is **divergent.**

4. [11 points] Zach is playing the retro video game *Plaque-Man* all day to get a new personal high score. Zach starts playing the game with 0 points. Over the course of each hour, Zach scores an additional 2500 points. At the **beginning** of every hour, Zach trades 20% of his points to buy extra time. For $n \geq 1$, let H_n be Zach's score at the **end** of the nth hour of playing the game. For example, $H_1 = 2500$.

a. [4 points] Write expressions for H_2 and H_3. Your answers should not involve the letter H. You do not need to simplify your expressions.

Solution: By definition, H_n is Zach's score at the end of the nth hour. At the beginning of the $(n+1)$th hour, Zach trades 20% of his points to buy extra time, so he has $0.8H_n$ points at the beginning of the $(n+1)$th hour. Over the course of the $(n+1)$th hour, Zach gets 2500 more points, so at the end of the $(n+1)$th hour, Zach's score is $H_{n+1} = 0.8H_n + 2500$. Since $H_1 = 2500$, we can use this recursive formula to write expressions for H_2 and H_3, shown below.

$H_2 =$ $2500(0.8) + 2500$

$H_3 =$ $2500(0.8)^2 + 2500(0.8) + 2500$

b. [4 points] Write a **closed-form** expression for H_n. *Closed-form* means your answer should not include ellipses (. . .) or sigma notation (Σ), and should not be recursive. You do not need to simplify your closed-form expression.

Solution: Continuing the pattern above from our expressions of H_1, H_2, and H_3, we find that

$$H_n = 2500(0.8)^{n-1} + \cdots + 2500(0.8)^2 + 2500(0.8) + 2500.$$

This is a finite geometric series. Recall the formula for the sum of a finite geometric series:

$$a + ax + ax^2 + \cdots + ax^{n-1} = \frac{a(1-x^n)}{1-x}, \qquad x \neq 1.$$

In the context of this problem, $a = 2500$ and $x = 0.8$, so we obtain our answer.

Answer: $H_n =$ $\dfrac{2500(1-0.8^n)}{1-0.8}$

c. [3 points] Find Zach's eventual score if he keeps playing *Plaque-Man* indefinitely. You do not need to simplify your numerical answer.

Solution: Zach's eventual score is found by taking a limit of the sequence H_n as $n \to \infty$:

$$\lim_{n\to\infty} H_n = \lim_{n\to\infty} \frac{2500(1-0.8^n)}{1-0.8} = \frac{2500}{1-0.8} = 12500.$$

Therefore Zach's eventual score will be 12500.

Answer: 12500

5. [14 points]

a. [7 points] Determine whether the following improper integral is convergent or divergent. **Fully justify** your answer including using **proper notation** and **showing mechanics** of any tests you use. You do not need to compute the value of the integral if it is convergent. Circle your final answer choice.

$$\int_1^\infty \frac{4+\sin(x)}{x^3+2}\,dx$$

Circle one: $\boxed{\textbf{Convergent}}$ **Divergent**

Solution: Note that 4 is larger than $\sin(x)$ for all x, and also x^3 dominates 2 for large x. Our intuition tells us that, for sufficiently large x,

$$\frac{4+\sin(x)}{x^3+2} \approx \frac{4}{x^3}.$$

Since $\int_1^\infty \frac{4}{x^3}\,dx$ converges, we expect the original improper integral to converge. We now need to make this reasoning rigorous.

Note that $-1 \le \sin(x) \le 1$ and $x^3+2 \ge x^3$ for all $x \ge 1$. This shows that

$$0 \le \frac{4+\sin(x)}{x^3+2} \le \frac{4+1}{x^3+2} \le \frac{4+1}{x^3} = \frac{5}{x^3} \qquad \text{for all } x \ge 1.$$

By the ***p*-test** ($p=3$), the integral $\int_1^\infty \frac{5}{x^3}\,dx$ is **convergent.**

Therefore, by the **comparison test,** the integral $\int_1^\infty \frac{4+\sin(x)}{x^3+2}\,dx$ is **convergent.**

b. [7 points] Let $0 < p < 1$ be a real number, and consider the improper integral

$$\int_1^3 \frac{1}{t(\ln(t))^p}\,dt.$$

The integral above converges; to show this, **compute** its value. Your answer may involve p. Be sure to show your full computation, and be sure to use **proper notation.**
Remember: $0 < p < 1$.

Solution: To evaluate this integral, we use proper notation and the substitution $w = \ln(t)$, so that $dw = \frac{1}{t}\,dt$, and so

$$\begin{aligned}\int_1^3 \frac{1}{t(\ln(t))^p}\,dt = \lim_{a\to 1^+}\int_a^3 \frac{1}{t(\ln(t))^p}\,dt &= \lim_{a\to 1^+}\int_{\ln(a)}^{\ln(3)} \frac{1}{w^p}\,dw \\ &= \lim_{a\to 1^+}\left(\frac{w^{1-p}}{1-p}\Big|_{\ln(a)}^{\ln(3)}\right) \\ &= \lim_{a\to 1^+}\left(\frac{\ln(3)^{1-p}}{1-p} - \frac{\ln(a)^{1-p}}{1-p}\right) = \frac{\ln(3)^{1-p}}{1-p}.\end{aligned}$$

(Note that $\ln(1)^{1-p} = 0$, as opposed to being undefined, since $0 < p < 1$ means that $1-p > 0$.)

Answer: $\int_1^3 \frac{1}{t(\ln(t))^p}\,dt =$ $\underline{\quad \frac{\ln(3)^{1-p}}{1-p} \quad}$

6. [10 points] Liban is writing songs using a new style of music which he calls "new-age jazz." The longer that he spends writing a particular song, the better it turns out.

Let $Q(t)$ be the **cumulative distribution function** (cdf) for t, the number of days that it takes for Liban to write a particular song. The formula for $Q(t)$ is shown to the right, where $c > 0$ is a constant.

$$Q(t) = \begin{cases} 0 & t < 0, \\ \dfrac{c}{4}t^2 & 0 \le t \le 2, \\ 2c - c\,e^{2-t} & t > 2. \end{cases}$$

You do not need to show your work in this problem, but partial credit may be given for work shown.

a. [3 points] Write a piecewise-defined formula for $q(t)$, the **probability density function** (pdf) corresponding to $Q(t)$. Your answer may involve c, but it should not involve the letter Q.

Solution: We know that $Q(t)$ and $q(t)$ are related by the formula $Q'(t) = q(t)$. So, the formula for $q(t)$ is found by taking the derivative of each part of $Q(t)$.

$$q(t) = \begin{cases} 0 & t < 0, \\ \dfrac{c}{2}t & 0 \le t \le 2, \\ c\,e^{2-t} & t > 2. \end{cases}$$

b. [3 points] Write an expression involving one or more integrals that represents the **mean** number of days that it takes for Liban to write a particular song. Your answer may involve c, but it should not involve the letters Q or q. **Do not evaluate your integral(s).**

Solution: The formula for the mean is given by $\int_{-\infty}^{\infty} tq(t)\,dt$. Using our answer to part (a):

$$\int_{-\infty}^{\infty} tq(t)\,dt = \int_{-\infty}^{0} 0\,dt + \int_0^2 \frac{c}{2}t^2\,dt + \int_2^{\infty} c\,te^{2-t}\,dt = \int_0^2 \frac{c}{2}t^2\,dt + \int_2^{\infty} c\,te^{2-t}\,dt.$$

Answer: $\displaystyle\int_0^2 \frac{c}{2}t^2\,dt + \int_2^{\infty} c\,te^{2-t}\,dt$

c. [2 points] Use the fact that $Q(t)$ is a cumulative distribution function to find the value of c.

Solution: Since $Q(t)$ is a cumulative distribution function, we must have $\lim\limits_{t\to\infty} Q(t) = 1$. Using the formula for $Q(t)$ for $t > 2$,

$$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty}(2c - c\,e^{2-t}) = 2c.$$

Therefore $2c = 1$, so that $c = \dfrac{1}{2}$.

Answer: $c = \dfrac{1}{2}$

d. [2 points] Circle the **one** correct answer below that completes the following sentence:

"The quantity $Q(5)$ *represents...*

(i) *...the probability that it takes exactly 5 days for Liban to write a song."*

(ii) *...the probability that it takes more than 5 days for Liban to write a song."*

$\boxed{\text{(iii)}}$ *...the probability that it takes 5 days or less for Liban to write a song."*

(iv) *...the approximate probability that it takes between 4.5 and 5.5 days for Liban to write a song."*

(v) NONE OF THESE

Solution: We know that $Q(5)$ and $q(5)$ are related by the formula $Q(5) = \int_{-\infty}^{5} q(t)\,dt$, and this integral represents the probability that it takes 5 days or less for Liban to write a song. To explain why the other choices are incorrect:

In general, the probability that it takes Liban between a and b days to write a song is the quantity

$$Q(b) - Q(a) = \int_a^b q(t)\,dt.$$

Thus the probability that it takes exactly 5 days for Liban to write a song must be

$$Q(5) - Q(5) = \int_5^5 q(t)\,dt = 0.$$

But $Q(5) \neq 0$ since we know $Q(t)$ is a nondecreasing function (as it is a cdf), and so we have $Q(5) \geq Q(2) = 0.5 > 0$, using our value of c from part (c). So (i) is incorrect.

The probability that it takes more than 5 days for Liban to write a song is $1 - Q(5) = \int_5^{\infty} q(t)\,dt$. We have $Q(2) = 0.5$. The formula for $Q(t)$ shows that it is strictly increasing, so $Q(5) > 0.5$, and thus $1 - Q(5) < 0.5$. This means $Q(5) \neq 1 - Q(5)$, so (ii) is incorrect.

The approximate probability that it takes between 4.5 and 5.5 days for Liban to write a song is a standard interpretation of the quantity $q(5)$, which describes a pdf, not a cdf. So (iv) is also incorrect.

7. [7 points] Determine whether the following series is convergent or divergent. **Fully justify** your answer including using **proper notation** and **showing mechanics** of any tests you use. Circle your final answer choice.

$$\sum_{n=2}^{\infty} \frac{4^n \cdot n^2}{n!}$$

Circle one: $\boxed{\textbf{Convergent}}$ **Divergent**

Solution: We claim that the series is convergent. To show this, we use the ratio test. Let $a_n = \frac{4^n \cdot n^2}{n!}$. Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\left|\frac{4^{n+1}\cdot(n+1)^2}{(n+1)!}\right|}{\left|\frac{4^n \cdot n^2}{n!}\right|} = \lim_{n\to\infty} \left|\frac{4^{n+1}\cdot(n+1)^2}{(n+1)!} \cdot \frac{n!}{4^n \cdot n^2}\right| = \lim_{n\to\infty} \frac{4(n+1)}{n^2} = 0 < 1.$$

Therefore, by the **ratio test,** the series $\sum_{n=2}^{\infty} \frac{4^n \cdot n^2}{n!}$ is **convergent.**

8. [6 points] Compute the following limit. **Fully justify** your answer including using **proper notation.**

$$\lim_{x\to 0} \frac{e^{2x} - (x+1)^2}{\cos(x) - 1}$$

Solution: We first note that this limit is of the indeterminate form $\frac{0}{0}$. We apply L'Hôpital's rule:

$$\lim_{x\to 0} \frac{e^{2x} - (x+1)^2}{\cos(x) - 1} \overset{L'H\ \frac{0}{0}}{=} \lim_{x\to 0} \frac{2e^{2x} - 2(x+1)}{-\sin(x)}.$$

But now this new limit is again of the indeterminate form $\frac{0}{0}$. We apply L'Hôpital's rule a second time:

$$\lim_{x\to 0} \frac{2e^{2x} - 2(x+1)}{-\sin(x)} \overset{L'H\ \frac{0}{0}}{=} \lim_{x\to 0} \frac{4e^{2x} - 2}{-\cos(x)} = \frac{4-2}{-1} = -2.$$

Therefore the original limit equals -2.

Answer: -2

9. [12 points] Determine whether the following series is absolutely convergent, conditionally convergent, or divergent. **Fully justify** your answer including using **proper notation** and **showing mechanics** of any tests you use. Circle your final answer choice.

$$\sum_{n=1}^{\infty} \frac{n \cdot (-1)^n}{n^2 + n^{1/2}}$$

Circle one: **Absolutely Convergent** $\boxed{\textbf{Conditionally Convergent}}$ **Divergent**

Solution: Observe that the given series is of the form

$$\sum_{n=1}^{\infty} (-1)^n a_n, \qquad \text{where} \qquad a_n = \frac{n}{n^2 + n^{1/2}}.$$

The sequence a_n satisfies $0 < a_{n+1} < a_n$ for all $n \geq 1$, and also $\lim_{n\to\infty} a_n = 0$.

Therefore, by the **alternating series test,** the series $\sum_{n=1}^{\infty} \frac{n \cdot (-1)^n}{n^2 + n^{1/2}}$ is **convergent.**

On the other hand, we claim that the original series is not absolutely convergent. To do this, we will show that the series

$$\sum_{n=1}^{\infty} \left| \frac{n \cdot (-1)^n}{n^2 + n^{1/2}} \right| = \sum_{n=1}^{\infty} \frac{n}{n^2 + n^{1/2}}$$

is divergent. There are two possible ways to do this:

Solution 1 (Direct comparison test)**:** Note that $n^{1/2} \leq n^2$ for $n \geq 1$, so that

$$\frac{n}{n^2 + n^{1/2}} \geq \frac{n}{n^2 + n^2} = \frac{1}{2n} \geq 0 \qquad \text{for all } n \geq 1.$$

By the ***p*-series test** $(p = 1)$**,** the series $\sum_{n=1}^{\infty} \frac{1}{2n}$ is **divergent.**

So, by the **direct comparison test,** the series $\sum_{n=1}^{\infty} \frac{n}{n^2 + n^{1/2}}$ is **divergent.**

Solution 2 (Limit comparison test)**:** Consider the following limit computation:

$$\lim_{n\to\infty} \frac{\frac{n}{n^2+n^{1/2}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2 + n^{1/2}} = \lim_{n\to\infty} \frac{1}{1 + \frac{1}{n^{3/2}}} = 1 > 0.$$

Note that this limit exists and is positive.

By the ***p*-series test** $(p = 1)$**,** the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is **divergent.**

So, by the **limit comparison test,** the series $\sum_{n=1}^{\infty} \frac{n}{n^2 + n^{1/2}}$ is **divergent.**

Therefore, the original series $\sum_{n=1}^{\infty} \frac{n \cdot (-1)^n}{n^2 + n^{1/2}}$ is **conditionally convergent.**

10. [6 points] Let $f(x)$, $g(x)$, and $h(x)$ be continuous functions defined for $x > 0$ which also satisfy the following:

- $\dfrac{1}{x^2} \le f(x) \le \dfrac{1}{x^3}$ for all $0 < x < 1$, and $0 \le f(x) \le \dfrac{1}{e^x}$ for all $x > 4$.
- $\displaystyle\sum_{n=1}^{\infty} g(n)$ is absolutely convergent.
- $h(x)$ is differentiable, increasing, and concave down, and $\displaystyle\lim_{x\to\infty} h(x) = 1$.

Determine whether the following integrals and series are **CONVERGENT** or **DIVERGENT**, and circle the appropriate answer. If there is not enough information to decide, circle **NEI**.
No justification is required.

a. [1 point] $\displaystyle\int_0^{\infty} f(x)\,dx$

Circle one: **CONVERGENT** $\boxed{\textbf{DIVERGENT}}$ **NEI**

Solution: We know that $f(x) \ge \dfrac{1}{x^2}$ for all $0 < x < 1$.

By the ***p*-test** $(p = 2)$**,** the integral $\displaystyle\int_0^1 \frac{1}{x^2}$ is **divergent.**

So, by the **comparison test,** the integral $\displaystyle\int_0^1 f(x)\,dx$ is **divergent.**

Thus, since $\displaystyle\int_0^{\infty} f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^{\infty} f(x)\,dx$, we conclude $\displaystyle\int_0^{\infty} f(x)\,dx$ is **divergent.**

b. [1 point] $\displaystyle\int_2^{\infty} f(x)\,dx$

Circle one: $\boxed{\textbf{CONVERGENT}}$ **DIVERGENT** **NEI**

Solution: We know that $0 \le f(x) \le \dfrac{1}{e^x} = e^{-x}$ for all $x > 4$.

By the **exponential decay test,** the integral $\displaystyle\int_4^{\infty} e^{-x}\,dx$ is **convergent.**

So, by the **comparison test,** the integral $\displaystyle\int_4^{\infty} f(x)\,dx$ is **convergent.**

Since $f(x)$ is continuous on $[2, 4]$, then $\displaystyle\int_2^4 f(x)\,dx$ is a proper integral, so it is **convergent.**

Thus, since $\displaystyle\int_2^{\infty} f(x)\,dx = \int_2^4 f(x)\,dx + \int_4^{\infty} f(x)\,dx$, we conclude $\displaystyle\int_2^{\infty} f(x)\,dx$ is **convergent.**

c. [1 point] $\displaystyle\int_1^\infty g(x)\,dx$

Circle one: **CONVERGENT** **DIVERGENT** $\boxed{\textbf{NEI}}$

Solution: If $g(x) = 0$, then $\displaystyle\sum_{n=1}^{\infty} |g(n)| = 0$, so $\displaystyle\sum_{n=1}^{\infty} g(n)$ is absolutely convergent. In this case, we also have $\displaystyle\int_1^\infty g(x)\,dx = 0$, so the integral is convergent.
On the other hand, consider the function $g(x) = \sin(\pi x)$. This function is continuous for $x > 0$ and, since $\sin(\pi n) = 0$ for all $n = 1, 2, 3, \ldots$, we have $\displaystyle\sum_{n=1}^{\infty} |g(n)| = 0$, so $\displaystyle\sum_{n=1}^{\infty} g(n)$ is again absolutely convergent. However,

$$\int_1^\infty g(x)\,dx = \int_1^\infty \sin(\pi x)\,dx = \lim_{b\to\infty}\int_1^b \sin(\pi x)\,dx = \lim_{b\to\infty}\left(-\frac{\cos(\pi b)}{\pi} + \frac{1}{\pi}\right) = \text{DNE}.$$

So the integral is divergent in this case.

d. [1 point] $\displaystyle\sum_{n=1}^{\infty} \frac{1}{g(n)}$

Circle one: **CONVERGENT** $\boxed{\textbf{DIVERGENT}}$ **NEI**

Solution: Since $\displaystyle\sum_{n=1}^{\infty} g(n)$ is absolutely convergent, it is convergent, and so $\displaystyle\lim_{n\to\infty} g(n) = 0$. This means $\displaystyle\lim_{n\to\infty}\frac{1}{g(n)} = \text{DNE}$.

By the ***n*th term test for divergence,** the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{g(n)}$ is **divergent.**

e. [1 point] $\displaystyle\sum_{n=1}^{\infty} h'(n)$

Circle one: $\boxed{\textbf{CONVERGENT}}$ **DIVERGENT** **NEI**

Solution: Since $h(x)$ is increasing and concave down, $h'(x)$ is positive and decreasing. Also,

$$\int_1^{\infty} h'(x)\,dx = \lim_{b\to\infty}\int_1^b h'(x)\,dx = \lim_{b\to\infty}(h(b)-h(1)) = 1-h(1).$$

By **direct computation,** we see that the integral $\displaystyle\int_1^{\infty} h'(x)\,dx$ is **convergent.**

Therefore, by the **integral test,** the series $\displaystyle\sum_{n=1}^{\infty} h'(n)$ is **convergent.**

f. [1 point] $\displaystyle\sum_{n=1}^{\infty} g(n)h(n)$

Circle one: $\boxed{\textbf{CONVERGENT}}$ **DIVERGENT** **NEI**

Solution: Since $\displaystyle\lim_{x\to\infty} h(x) = 1$, then the sequence $h(n)$ is convergent, hence bounded. So, there is some number $M > 0$ such that $|h(n)| \le M$ for all $n = 1, 2, 3, \ldots$. Now we see that

$$|g(n)h(n)| = |g(n)|\cdot|h(n)| \le M|g(n)| \qquad \text{for all } n \ge 1.$$

Since $\displaystyle\sum_{n=1}^{\infty} g(n)$ is **absolutely convergent,** then $\displaystyle\sum_{n=1}^{\infty} M|g(n)|$ is **convergent.**

By the **comparison test,** the series $\displaystyle\sum_{n=1}^{\infty} |g(n)h(n)|$ is **convergent.**

Therefore $\displaystyle\sum_{n=1}^{\infty} g(n)h(n)$ is **absolutely convergent** (hence convergent)**.**