Math 116 — Final Exam — April 25, 2024

EXAM SOLUTIONS

1. This exam has 17 pages including this cover.
2. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. The last page of this exam is a formula sheet which you may remove. Please submit that page along with your exam. Work written on that page will not be graded.
4. Do not separate the other pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
7. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
8. You are allowed notes written on two sides of a $3'' \times 5''$ notecard. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
9. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
10. Include units in your answer where that is appropriate.
11. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
12. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	12	
2	12	
3	9	
4	10	
5	12	

Problem	Points	Score
6	10	
7	8	
8	8	
9	9	
10	10	
Total	100	

1. [12 points] Let $g(x)$ be a **differentiable** function, and let $G(x)$ be a **continuous antiderivative** of $g(x)$. Some values of $g(x)$ and $G(x)$ are given in the table below:

x	-2	-1	0	1	2
$g(x)$	0	$\sqrt{3}$	4	5	-1
$G(x)$	π	$1/2$	-2	0	1

Use the table above to answer the following questions. Write your answers in **exact form**. If there is not enough information to complete a problem, write "NEI." Your answers should not involve the letters g or G, but you do not need to simplify your final answers. Show all your work.

a. [3 points] Compute the **average value** of $g'\big(g(x)\big) \cdot g'(x)$ on the interval $[-2, 2]$.

Solution: We use a substitution $u = g(x)$, so that $du = g'(x)\,dx$, and so the average value is

$$\frac{1}{2-(-2)}\int_{-2}^{2} g'\big(g(x)\big)\cdot g'(x)\,dx \;=\; \frac{1}{4}\int_{0}^{-1} g'(u)\,du \;=\; \frac{1}{4}\big(g(-1)-g(0)\big) \;=\; \frac{1}{4}(\sqrt{3}-4).$$

Answer: $\frac{1}{4}(\sqrt{3}-4)$

b. [3 points] Compute $F'(1)$, where $F(x) = \displaystyle\int_{x^3-2}^{4} G(t)\,dt$.

Solution: By the Second Fundamental Theorem of Calculus and the chain rule,

$$F'(x) = -3x^2G(x^3-2), \qquad \text{and so} \qquad F'(1) = -3G(-1) = -3/2.$$

Answer: $-\frac{3}{2}$

c. [3 points] Approximate $\displaystyle\int_{-2}^{2} G(x)\,dx$ using TRAP(2).

Solution: The subintervals we use are $[-2, 0]$ and $[0, 2]$. The formula for TRAP(2) is thus

$$\text{TRAP}(2) \;=\; \frac{1}{2}\cdot 2\big(G(-2)+2G(0)+G(2)\big) \;=\; \pi + 2(-2) + 1 \;=\; \pi - 3.$$

Answer: $\pi - 3$

d. [3 points] Compute $\displaystyle\lim_{x\to\infty} x\,G(1+\tfrac{1}{x})$.

Solution: Since $\displaystyle\lim_{x\to\infty} G(1+\tfrac{1}{x}) = G(1) = 0$, the given limit is of the indeterminate form $\infty \cdot 0$. Recalling that $G'(x) = g(x)$, we rearrange and apply L'Hôpital's rule:

$$\lim_{x\to\infty} x\,G(1+\tfrac{1}{x}) \;=\; \lim_{x\to\infty}\frac{G(1+\frac{1}{x})}{\frac{1}{x}} \;\overset{L'H\,\frac{0}{0}}{=}\; \lim_{x\to\infty}\frac{-\frac{1}{x^2}g(1+\frac{1}{x})}{-\frac{1}{x^2}} \;=\; \lim_{x\to\infty} g(1+\tfrac{1}{x}) \;=\; g(1) \;=\; 5.$$

Answer: 5

2. [12 points] In the video game *Super Maria 64*, sisters Maria and Luisa travel through the Lilypad Kingdom to collect magical Rainbow Crystals. In the Sandland Desert, represented by the xy-plane, the sisters run around collecting all the Rainbow Crystals they see.
All distances in this problem are measured in kilometers. For $t \geq 0$, the sisters' positions t hours after they start running are given by the following parametric equations:

$$\text{Maria: } \begin{cases} x(t) = t^2 + t - 6 \\ y(t) = 2\sin(\pi t) \end{cases} \qquad \text{Luisa: } \begin{cases} x(t) = 2t^2 - 4t \\ y(t) = \cos(\frac{\pi}{2}t) \end{cases}$$

a. [2 points] Find **Maria's position** 1 hour after the sisters start running.

Solution: Maria's (x, y) position at $t = 1$ is given by $(x(1), y(1))$, where for Maria, $x(1) = -4$ and $y(1) = 0$.

Answer: $x =$ -4 $y =$ 0

b. [3 points] Find **Maria's speed,** in kilometers per hour, 1 hour after the sisters start running.

Solution: Maria's instantaneous speed at $t = 1$ is given by $\sqrt{x'(1)^2 + y'(1)^2}$, where for Maria, $x'(t) = 2t + 1$ and $y'(t) = 2\pi\cos(\pi t)$, so $x'(1) = 3$ and $y'(1) = -2\pi$.

Answer: $\sqrt{3^2 + (-2\pi)^2}$

c. [3 points] Find **all** times $t \geq 0$ at which **Luisa** travels **directly north** (that is, not in any northwest or northeast direction). If there is no such time, write "NONE." Show your work to justify your answer.

Solution: In order for Luisa to travel directly north, we must have $x'(t) = 0$ and $y'(t) > 0$, where for Luisa, $x'(t) = 4t - 4$ and $y'(t) = -\frac{\pi}{2}\sin(\frac{\pi}{2}t)$. The only value of t that satisfies $x'(t) = 0$ is $t = 1$. However, $y'(1) = -\frac{\pi}{2} < 0$, so Luisa travels directly **south** at the time $t = 1$. Therefore there are no times t where Luisa travels directly north.

Answer: $t =$ **NONE**

d. [4 points] Find **all** times $t \geq 0$ at which Maria and Luisa are at the same position. If there is no such time, write "NONE." Show your work to justify your answer.

Solution: The sisters are at the same x-position when $t^2 + t - 6 = 2t^2 - 4t$. We solve for t:

$$\begin{aligned} t^2 + t - 6 &= 2t^2 - 4t \\ 0 &= t^2 - 5t + 6 \\ 0 &= (t-2)(t-3) \end{aligned}$$

So, Maria and Luisa are at the same x-position when $t = 2$ and $t = 3$. However, observe that Maria and Luisa are not at the same y-position when $t = 2$ since $2\sin(2\pi) \neq \cos(\frac{\pi}{2} \cdot 2)$. On the other hand, they are at the same y-position at $t = 3$ since $2\sin(3\pi) = \cos(\frac{\pi}{2} \cdot 3)$. So $t = 3$ is the only time at which they are at the same position.

Answer: $t =$ 3

3. [9 points] The Taylor series centered at $x = 1$ for a function $T(x)$ is given by:

$$T(x) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(-5)^n \cdot (2n)!} (x-1)^{4n+3}.$$

a. [6 points] Find the **radius of convergence** of the Taylor series above. Show your work. Do not attempt to find the interval of convergence.

Solution: We use the ratio test: letting $a_n = \dfrac{(n!)^2(x-1)^{4n+3}}{(-5)^n \cdot (2n)!}$, we have

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{((n+1)!)^2(x-1)^{4n+7}}{(-5)^{n+1}\cdot(2n+2)!} \cdot \frac{(-5)^n\cdot(2n)!}{(n!)^2(x-1)^{4n+3}}\right| \\
&= \lim_{n\to\infty} \frac{(n+1)(n+1)|x-1|^4}{5(2n+2)(2n+1)} \\
&= \frac{|x-1|^4}{20}
\end{aligned}$$

The series converges when $\dfrac{|x-1|^4}{20} < 1$; that is, when $|x-1|^4 < 20$, and so $|x-1| < 20^{1/4}$. Therefore the radius of convergence is $20^{1/4}$.

Answer: $20^{1/4}$

b. [3 points] Compute $T^{(123)}(1)$. Show your work. You do not need to simplify your answer.

Solution: The quantity $\dfrac{T^{(123)}(1)}{123!}$ is the coefficient of $(x-1)^{123}$ in the Taylor series for $T(x)$. Using the formula for the Taylor series of $T(x)$, we must find n such that

$$\frac{T^{(123)}(1)}{123!}(x-1)^{123} = \frac{(n!)^2}{(-5)^n \cdot (2n)!}(x-1)^{4n+3}.$$

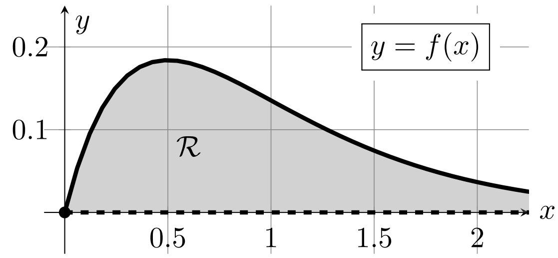
Comparing powers of $x-1$, we solve $123 = 4n+3$ to get $n = 30$, and so we obtain

$$\frac{T^{(123)}(1)}{123!} = \frac{(30!)^2}{(-5)^{30}\cdot 60!}.$$

Now we multiply both sides by 123! to get the answer.

Answer: $T^{(123)}(1) = \dfrac{(30!)^2 \cdot 123!}{(-5)^{30}\cdot 60!}$

4. [10 points] Louise, the world-famous abstract artist and cheese enthusiast, has a dream about an infinitely-long cheese sculpture. The sculpture involves $\mathcal{R}$, which is the region above the x-axis and below the curve $f(x) = xe^{-2x}$ on the interval $[0, \infty)$. A portion of $\mathcal{R}$ is the shaded region below.



a. [4 points] Write an improper integral that represents the **volume** of the infinitely-long solid of revolution formed by rotating the region $\mathcal{R}$ around the x-axis.
Your answer should not involve the letter f. **Do not evaluate your integral.**

Solution: We use the disk method (the washer method but with no inner radius). The disk method states that the volume of a disk of thickness Δx at a point $x \geq 0$ is of the form $\pi R(x)^2 \Delta x$ for some function $R(x)$ representing the radius of our disk. For each $x \geq 0$, the radius $R(x)$ is given by the distance from the x-axis to $y = f(x)$, which is $R(x) = xe^{-2x}$. This gives us our answer:

$$\int_0^\infty \pi \left(xe^{-2x}\right)^2 dx$$

Note: To use the shell method, as we are rotating around a horizontal axis, we would need to write an integral over $0 \leq y \leq \frac{1}{2}e^{-1}$*, where* $\frac{1}{2}e^{-1}$ *is the global maximum of* $f(x)$ *on the interval* $[0, \infty)$*. This requires us to solve the equation* $y = f(x)$ *for* x *in terms of* y*. But we cannot solve the equation* $y = xe^{-2x}$ *for* x*, so we cannot write an integral using the shell method here.*

Answer: $\int_0^\infty \pi \left(xe^{-2x}\right)^2 dx$

b. [6 points] The **area** of the region $\mathcal{R}$ (not the volume of the rotated solid) is given by the improper integral

$$\int_0^\infty xe^{-2x}\,dx.$$

Determine whether this improper integral is convergent or divergent.
You may use either a direct computation or the comparison test to reach your conclusion.
Fully justify your answer including using **proper notation.** Circle your final answer choice.

Circle one: **Convergent** **Divergent**

Solution: There are two ways to solve this problem:

Solution 1 (Direct computation)**:** We can directly evaluate this integral. To do so, we use proper notation and integrate by parts:

$$\begin{aligned}
\int_0^\infty xe^{-2x}\,dx &= \lim_{b\to\infty}\int_0^b xe^{-2x}\,dx \\
&= \lim_{b\to\infty}\left(-\frac{1}{2}xe^{-2x}\Big|_0^b + \frac{1}{2}\int_0^b e^{-2x}\,dx\right) \\
&= \lim_{b\to\infty}\left(-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\Big|_0^b\right) \\
&= \lim_{b\to\infty}\left(-\frac{1}{2}be^{-2b} - \frac{1}{4}e^{-2b} + 0 + \frac{1}{4}\right) = \frac{1}{4}
\end{aligned}$$

(The limit $\lim\limits_{b\to\infty} be^{-2b}$ can be computed by L'Hôpital's rule or a dominating functions argument.)
Therefore, by **direct computation,** the integral is **convergent.**

Solution 2 (Comparison test)**:** Alternatively, we can show that the integral is convergent without computing its value. Observe that $x \le e^x$ for all $x \ge 0$, so that

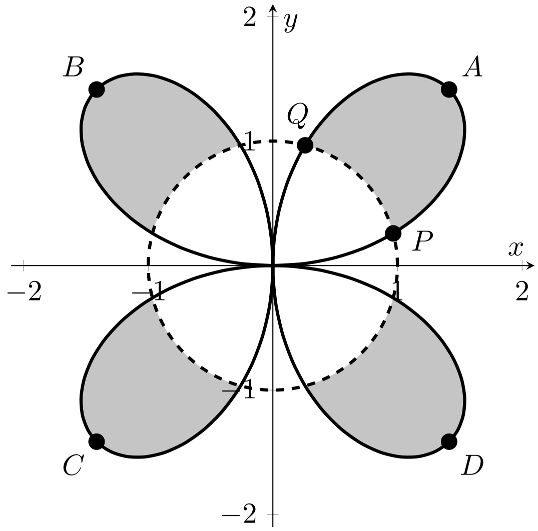
$$xe^{-2x} \le e^x e^{-2x} = e^{-x} \qquad \text{for all } x \ge 0.$$

By the **exponential decay test,** the integral $\displaystyle\int_0^\infty e^{-x}\,dx$ is **convergent.**

Therefore, by the **comparison test,** the integral $\displaystyle\int_0^\infty xe^{-2x}\,dx$ is **convergent.**

5. [12 points] A large four-leaf clover, pictured below, resides in a forest.

- The leaves of the clover are modeled by the polar curve
$$r = 2\sin(2\theta)$$
for $0 \le \theta \le 2\pi$. This is the **solid** curve in the diagram to the right.
- The polar curve
$$r = 1$$
for $0 \le \theta \le 2\pi$ is the **dashed** curve in the diagram to the right.
The leaves of the clover are light green inside of this curve, and dark green outside of it.
- All distances are measured in inches.



a. [2 points] Which of the following points labelled in the diagram above is in the portion of the polar curve $r = 2\sin(2\theta)$ traced out for $\frac{\pi}{2} \le \theta \le \pi$? Circle the **one** correct answer.
No justification is required.

Circle one: A B C $\boxed{D}$ NONE OF THESE

Solution: We substitute $\theta = \frac{3\pi}{4}$ (which satisfies $\frac{\pi}{2} \le \theta \le \pi$) into $r = 2\sin(2\theta)$ to get $r = 2\sin(\frac{3\pi}{2}) = -2$. So, the relevant polar coordinate is $(r, \theta) = (-2, \frac{3\pi}{4})$. By converting this into Cartesian coordinates using the formulas $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we get the point $(x, y) = (-2\cos(\frac{3\pi}{4}), -2\sin(\frac{3\pi}{4}))$, that is, $(x, y) = (\frac{2}{\sqrt{2}}, -\frac{2}{\sqrt{2}})$. This is the point D.

b. [5 points] The points P and Q, labelled above, are two intersection points of the solid and dashed curves. Write P and Q in **polar coordinates** $\boldsymbol{(r, \theta)}$**,** where $r \ge 0$ and $0 \le \theta \le 2\pi$.
Please show all of your work.

Solution: The polar coordinates (r, θ) of P and Q are both of the form $(1, \theta)$ since P and Q are on the circle $r = 1$, so we just need to find the θ-values. We must solve $2\sin(2\theta) = 1$ for θ, that is, $\sin(2\theta) = \frac{1}{2}$. We note that $\sin(x) = \frac{1}{2}$ whenever $x = \frac{\pi}{6} + 2\pi k$ or $x = \frac{5\pi}{6} + 2\pi k$ for an integer k. Thus $\sin(2\theta) = \frac{1}{2}$ gives the following:

$$2\theta = \frac{\pi}{6} + 2\pi k \quad \Rightarrow \quad \theta = \frac{\pi}{12} + \pi k \quad \Rightarrow \quad \theta = \frac{\pi}{12}, \frac{13\pi}{12},$$

$$2\theta = \frac{5\pi}{6} + 2\pi k \quad \Rightarrow \quad \theta = \frac{5\pi}{12} + \pi k \quad \Rightarrow \quad \theta = \frac{5\pi}{12}, \frac{17\pi}{12}.$$

The θ-values listed above are the ones which satisfy $0 \le \theta \le 2\pi$. But, since P and Q are in the first quadrant and $r > 0$, we see that P must correspond with $\theta = \frac{\pi}{12}$, and Q must correspond with $\theta = \frac{5\pi}{12}$.

Answer: P: $(r, \theta) =$ $(1, \frac{\pi}{12})$ Q: $(r, \theta) =$ $(1, \frac{5\pi}{12})$

c. [5 points] Write an expression involving **at most two integrals** that gives the **area**, in square inches, of the dark green part of the four-leaf clover. (This is the shaded region in the diagram above.) **Do not evaluate your integral(s).**

Solution: A useful formula for the area bounded by a polar curve $r = f(\theta)$ is

$$\frac{1}{2}\int_{\alpha}^{\beta} f(\theta)^2\, d\theta.$$

The points P and Q are relevant: using our θ-values from part (b), the area formula with $f(\theta) = 2\sin(2\theta)$ gives the integral

$$\frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} (2\sin(2\theta))^2\, d\theta, \qquad (*)$$

which is the area of the shaded region shown below to the left. We need to remove the extra area inside the dashed circle $r = 1$, shown below in the middle. Here are two possible ways to do this:

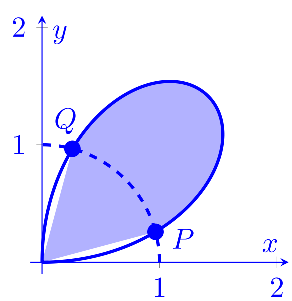
Solution 1 (Area formula)**:** The extra area inside the dashed circle can can be found using the area formula with $f(\theta) = 1$: its area is $\frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} 1^2\, d\theta$. Subtracting this from (*), we get

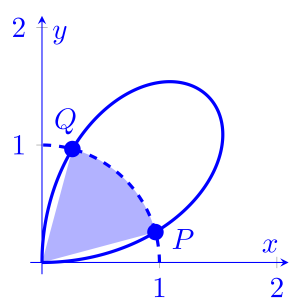
$$\frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} (2\sin(2\theta))^2\, d\theta - \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} 1^2\, d\theta.$$

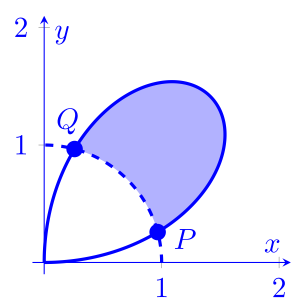
Solution 2 (Area of a circle)**:** Note that the area inside the dashed circle is one-sixth the area of a circle of radius 1 (since $\frac{5\pi}{12} - \frac{\pi}{12} = \frac{2\pi}{6}$), hence has area $\frac{\pi}{6}$. Subtracting this from (*), we get

$$\frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} (2\sin(2\theta))^2\, d\theta - \frac{\pi}{6}.$$

Either way, we now have the area of the shaded region shown below to the right. Finally, since this is only the area on one leaf of the four-leaf clover, we multiply our expression by 4, using the symmetry of the clover, to arrive at our final answer.



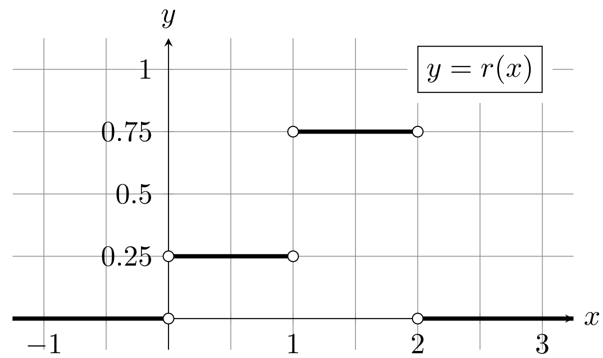




Answer: $2\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} (2\sin(2\theta))^2\, d\theta - 2\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} 1^2\, d\theta$ **OR** $2\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} (2\sin(2\theta))^2\, d\theta - \frac{2\pi}{3}$

6. [10 points]
A survey has recently been conducted on the University of Michigan campus which asked a large number of students to choose a random real number in the interval $[0, 2]$.

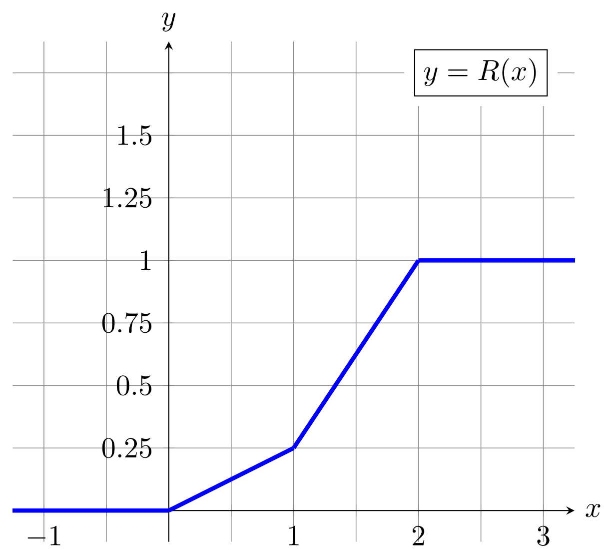
The numbers chosen by students are described by the **probability density function** (pdf) $r(x)$. A graph of $r(x)$ is shown to the right.



You do not need to show your work in this problem, but partial credit may be given for work shown.

a. [5 points]
Let $R(x)$ be the **cumulative distribution function** (cdf) corresponding to $r(x)$.
The function $R(x)$ is defined for all real numbers x.
On the axes provided to the right, sketch a graph of $R(x)$ **on the interval $[-1, 3]$.**
Be sure to pay attention to:

- where $R(x)$ is and is not differentiable;
- where $R(x)$ is increasing, decreasing, or constant;
- where $R(x)$ is concave up, concave down, or linear;
- the values of $R(x)$ at $x = -1, 0, 1, 2,$ and 3.



b. [2 points] Compute the fraction of students that chose a number in the interval $[1, 2]$.

Solution: There are two ways to solve this problem:

Solution 1 (Using the pdf $r(x)$)**:** The answer is given by the quantity $\int_1^2 r(x)\,dx$. Using the graph of $r(x)$, we have $\int_1^2 r(x)\,dx = \int_1^2 0.75\,dx = 0.75$.

Solution 2 (Using the cdf $R(x)$)**:** The answer is given by the quantity $R(2) - R(1)$. Using our graph of $R(x)$ from part (a), we have $R(2) - R(1) = 1 - 0.25 = 0.75$.

Answer: 3/4 **or** 0.75 **or** 75%

c. [3 points] Compute the **median** of the numbers chosen among all students.

Solution: There are two ways to solve this problem:

Solution 1 (Using the pdf $r(x)$)**:** The median value of x is the number T such that $\int_{-\infty}^{T} r(x)\,dx = 0.5$, or equivalently, $\int_{T}^{\infty} r(x)\,dx = 0.5$. Using the graph of $r(x)$, since

$$\int_{1}^{\infty} r(x)\,dx = \int_{1}^{2} 0.75\,dx = 0.75,$$

we see that we must have $1 < T < 2$. It follows that in the graph of $r(x)$, we are looking for the value of T so that the rectangle with base $[T, 2]$ and height 0.75 has area 0.5. Or, using integrals, we have

$$0.5 = \int_{T}^{\infty} r(x)\,dx = \int_{T}^{2} 0.75\,dx = 0.75(2 - T).$$

We solve the equation $0.5 = 0.75(2 - T)$ for T, which gives $T = 4/3$. Therefore the median value of x is $T = 4/3$.

Solution 2 (Using the cdf $R(x)$)**:** The median value of x is the number T such that $R(T) = 0.5$. Using our graph of $R(x)$ from part (a), we see that $1 < T < 2$. For such T, the slope of $R(x)$ at $x = T$ is $3/4$, so by using the slope formula, we set up an equation involving T and solve:

$$\frac{0.5 - 0.25}{T - 1} = \frac{3}{4} \quad \Rightarrow \quad 4(0.5 - 0.25) = 3(T - 1) \quad \Rightarrow \quad 2 - 1 = 3T - 3 \quad \Rightarrow \quad T = \frac{4}{3}.$$

Therefore the median value of x is $T = 4/3$.

Answer: 4/3

7. [8 points] Consider the power series below, centered at $x = 2$:

$$\sum_{n=1}^{\infty} \frac{n+2}{4^n \cdot n^3} (x-2)^n.$$

Its radius of convergence is 4; you do not need to show this.
Find the **interval of convergence** of this power series. Fully justify your answer including using proper notation and showing mechanics of any tests you use.

Solution: The center of convergence for this power series is 2. Since its radius of convergence is 4, we only need to check convergence at the endpoints $2 \pm 4 = -2, 6$.

At $x = 6$, the series is

$$\sum_{n=1}^{\infty} \frac{n+2}{4^n \cdot n^3} (6-2)^n = \sum_{n=1}^{\infty} \frac{n+2}{4^n \cdot n^3} 4^n = \sum_{n=1}^{\infty} \frac{n+2}{n^3}.$$

Consider the following limit computation:

$$\lim_{n\to\infty} \frac{\frac{n+2}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3+2n^2}{n^3} = \lim_{n\to\infty} \frac{1+\frac{2}{n}}{1} = 1 > 0.$$

Note that this limit exists and is positive.
By the ***p*-series test** ($p = 2$)**,** the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is **convergent.**
So, by the **limit comparison test,** the series $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{n^3}$ is **convergent.**
Therefore $x = 6$ is included in our interval of convergence.

At $x = -2$, the series is

$$\sum_{n=1}^{\infty} \frac{n+2}{4^n \cdot n^3} (-2-2)^n = \sum_{n=1}^{\infty} \frac{n+2}{4^n \cdot n^3} (-4)^n = \sum_{n=1}^{\infty} \frac{n+2}{4^n \cdot n^3} (-1)^n 4^n = \sum_{n=1}^{\infty} \frac{n+2}{n^3} (-1)^n.$$

Now observe that

$$\sum_{n=1}^{\infty} \left| \frac{n+2}{n^3} (-1)^n \right| = \sum_{n=1}^{\infty} \frac{n+2}{n^3}.$$

Above, we showed that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{n^3}$ is **convergent.**
So, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{n^3} (-1)^n$ is **absolutely convergent** (hence convergent)**.**
Therefore $x = -2$ is included in our interval of convergence.

We conclude that the interval of convergence is $[-2, 6]$.

Solution: We present an **alternate solution** using different tests to prove convergence. As a reminder, the center of convergence for this power series is 2. Since its radius of convergence is 4, we only need to check convergence at the endpoints $2 \pm 4 = -2, 6$.

At $x = 6$, the series is

$$\sum_{n=1}^{\infty} \frac{n+2}{4^n \cdot n^3} (6-2)^n = \sum_{n=1}^{\infty} \frac{n+2}{4^n \cdot n^3} 4^n = \sum_{n=1}^{\infty} \frac{n+2}{n^3}.$$

Note that $n + 2 \leq n + n = 2n$ for $n \geq 2$, so that

$$\frac{n+2}{n^3} \leq \frac{n+n}{n^3} \leq \frac{2n}{n^3} = \frac{2}{n^2} \qquad \text{for all } n \geq 2.$$

By the ***p*-series test** $(p = 2)$**,** the series $\sum_{n=1}^{\infty} \frac{2}{n^2}$ is **convergent.**

So, by the **direct comparison test,** the series $\sum_{n=1}^{\infty} \frac{n+2}{n^3}$ is **convergent.**
Therefore $x = 6$ is included in our interval of convergence.

At $x = -2$, the series is

$$\sum_{n=1}^{\infty} \frac{n+2}{4^n \cdot n^3} (-2-2)^n = \sum_{n=1}^{\infty} \frac{n+2}{4^n \cdot n^3} (-4)^n = \sum_{n=1}^{\infty} \frac{n+2}{4^n \cdot n^3} (-1)^n 4^n = \sum_{n=1}^{\infty} \frac{n+2}{n^3} (-1)^n.$$

Observe that this series is of the form

$$\sum_{n=1}^{\infty} (-1)^n a_n, \qquad \text{where} \qquad a_n = \frac{n+2}{n^3}.$$

The sequence a_n satisfies $0 < a_{n+1} < a_n$ for all $n \geq 1$, and also $\lim_{n \to \infty} a_n = 0$.

So, by the **alternating series test,** the series $\sum_{n=1}^{\infty} \frac{n+2}{n^3} (-1)^n$ is **convergent.**
Therefore $x = -2$ is included in our interval of convergence.

We once again conclude that the interval of convergence is $[-2, 6]$.

Answer: $[-2, 6]$

8. [8 points]

a. [4 points] Write down the first 3 nonzero terms of the Taylor series for the function

$$S(x) = \begin{cases} \dfrac{e^{x^2} - 1}{3x^2} & x \neq 0, \\ \dfrac{1}{3} & x = 0, \end{cases}$$

centered at $x = 0$. You do not need to simplify any numbers in your answer.

Solution: We can use the "known" Taylor series for e^x:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ e^{x^2} &= 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \\ e^{x^2} - 1 &= x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \\ \frac{e^{x^2} - 1}{3x^2} &= \frac{1}{3} + \frac{x^2}{3 \cdot 2!} + \frac{x^4}{3 \cdot 3!} + \cdots \end{aligned}$$

This gives the first three nonzero terms.

Answer: $\frac{1}{3} + \frac{x^2}{3 \cdot 2!} + \frac{x^4}{3 \cdot 3!}$

b. [4 points] Compute the following limit. **Fully justify** your answer including using **proper notation.**

$$\lim_{x \to 0} \frac{\int_0^x \left(e^{t^2} - 1\right) dt}{x^3}$$

Hint: Your answer from the previous part may be helpful at some point.

Solution: We start by using L'Hôpital's rule. To take the derivative of the numerator, we apply the Second Fundamental Theorem of Calculus (note that the integral cannot be evaluated directly):

$$\lim_{x \to 0} \frac{\int_0^x \left(e^{t^2} - 1\right) dt}{x^3} \overset{L'H\,\frac{0}{0}}{=} \lim_{x \to 0} \frac{e^{x^2} - 1}{3x^2}.$$

Below are two possible ways to complete the problem:

Solution 1 (Taylor series)**:** Here we use our Taylor series from part (a). From this, we have

$$\lim_{x \to 0} \frac{e^{x^2} - 1}{3x^2} = \lim_{x \to 0} \left(\frac{1}{3} + \frac{x^2}{3 \cdot 2!} + \frac{x^4}{3 \cdot 3!} + \cdots \right) = \frac{1}{3}.$$

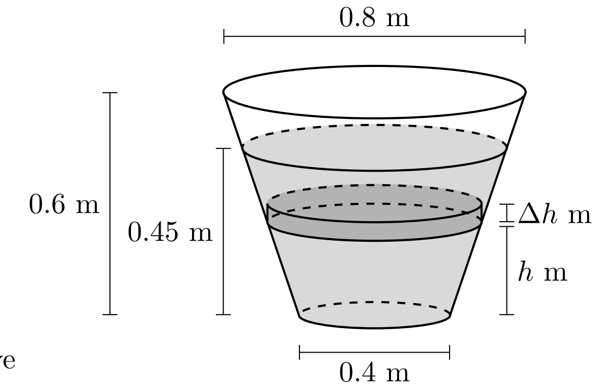
Solution 2 (L'Hôpital's rule)**:** Alternatively, we can apply L'Hôpital's rule again:

$$\lim_{x \to 0} \frac{e^{x^2} - 1}{3x^2} \overset{L'H\,\frac{0}{0}}{=} \lim_{x \to 0} \frac{2xe^{x^2}}{6x} = \lim_{x \to 0} \frac{e^{x^2}}{3} = \frac{1}{3}$$

Answer: $\frac{1}{3}$

9. [9 points] Over the course of the summer, Markie makes many pitchers of their favorite lemonade. The special ingredient is a mixture of different types of sugar, which Markie scoops out of the extremely large bowl pictured below.

- The top of the bowl is circular and has a **diameter** of 0.8 meters.
- The bottom of the bowl is circular and has a **diameter** of 0.4 meters.
- The height of the bowl is 0.6 meters.
- Initially, the bowl is only filled up to 0.45 meters from the bottom of the bowl.
- The sugar mixture has an uneven density of $100(9-4h)$ kg/m^3, where h is the distance above the bottom of the bowl, in meters.



As suggested by the picture above, the diameter of a circular cross-section is a linear function of h. In this problem, you may assume that the acceleration due to gravity is $g = 9.8$ m/s^2.

a. [5 points] Write an expression that approximates the **mass,** in kilograms, of **a slice** of sugar that is h meters above the bottom of the bowl and has small thickness Δh meters (as shaded in the picture above). Your answer should not involve any integrals.

Solution: First we must find the radius $r(h)$ of a circular cross-section which is h meters above the bottom of the bowl. When $h = 0$ the radius is 0.2, and when $h = 0.6$ the radius is 0.4. Since r increases at a constant rate with respect to h, the slope of the resulting linear equation for $r(h)$ is $\frac{0.4-0.2}{0.6-0} = \frac{1}{3}$. Therefore $r(h) = \frac{1}{3}h + 0.2$.
The area of a circular cross-section that is h meters above the bottom of the bowl is given by $\pi r(h)^2 = \pi(\frac{1}{3}h + 0.2)^2$ m^2. The approximate volume of a slice is thus $\pi(\frac{1}{3}h+0.2)^2\,\Delta h$ m^3. Therefore the approximate mass of a slice is given by $100\pi(\frac{1}{3}h+0.2)^2(9-4h)\,\Delta h$ kg.

Answer: $100\pi\left(\frac{1}{3}h+0.2\right)^2(9-4h)\,\Delta h$

b. [4 points]

(i) Write an expression that approximates the **amount of work,** in Joules, needed to lift **a slice** of sugar that is h meters above the bottom of the bowl and has small thickness Δh meters (as shaded in the picture above) to the **top of the bowl.**
Your answer should not involve any integrals.

Solution: The approximate amount of work needed is (work) = (force)(distance). The slice is lifted from h meters above the bottom of the bowl to the top, so it is lifted a distance of $0.6-h$ meters. The force required is (force) = (mass)(acceleration); the mass is our answer from part (a), and the acceleration is $g = 9.8$ m/s^2. Putting this all together, we obtain our answer.

Answer: $9800\pi\left(\frac{1}{3}h+0.2\right)^2(9-4h)(0.6-h)\,\Delta h$

(ii) Write an expression involving an integral representing the **total amount of work,** in Joules, needed for Markie to lift **all** the sugar to the **top of the bowl.**

Solution: The total amount of work is obtained by taking the integral of the amount of work, from part (i), that it takes to lift all slices as $\Delta h \to 0$ for $0 \le h \le 0.45$. This gives our answer in the form of an integral.

Answer: $\displaystyle\int_0^{0.45} 9800\pi\left(\frac{1}{3}h+0.2\right)^2(9-4h)(0.6-h)\,dh$

10. [10 points] For each of the following parts, circle **all** correct answers. No justification is required.

a. [2 points] A power series $\sum_{n=0}^{\infty} A_n(x+1)^n$ **converges** at $x = -3$ and **diverges** at $x = 2$.

At which of the following x-values, if any, **must** this power series **converge?**

$x = -4$ $\boxed{x = -2}$ $\boxed{x = 0}$ $x = 1$ $x = 3$ NONE

Solution: The center of convergence is $x = -1$. Since the power series converges at $x = -3$, its radius of convergence must be at least 2, so its interval of convergence must contain the interval $[-3, 1)$. So the power series **must** converge at $x = -2$ and $x = 0$. Of the x-values listed, there are no others where the power series **must** converge, because $[-3, 1)$ could be the exact interval of convergence of this power series.

b. [2 points] Another power series $\sum_{n=0}^{\infty} B_n(x+1)^n$ **converges** for all $x < -3$.

At which of the following x-values, if any, **must** this power series **converge?**

$\boxed{x = -4}$ $\boxed{x = -2}$ $\boxed{x = 0}$ $\boxed{x = 1}$ $\boxed{x = 3}$ NONE

Solution: If a power series converges for all $x < -3$, then its interval of convergence must contain the interval $(-\infty, -3)$. But then the radius of convergence must be ∞, so the power series must in fact converge for all $x \geq -3$, too.

c. [2 points] Which, if any, of the following infinite series **converge** to $\frac{1}{2}$?

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ $\quad$ $\boxed{\displaystyle\sum_{n=0}^{\infty} \frac{3}{8}\left(\frac{1}{4}\right)^n}$ $\quad$ $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2}{2n^2}$ $\quad$ $\boxed{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\pi}{3}\right)^{2n}}$ $\quad$ NONE

Solution: First, using the "known" Taylor series for $\ln(1+x)$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot 1^n = \ln(1+1) = \ln(2) \neq \frac{1}{2}.$$

Next, using the formula for the sum of an infinite geometric series, we have

$$\sum_{n=0}^{\infty} \frac{3}{8}\left(\frac{1}{4}\right)^n = \frac{\frac{3}{8}}{1-\frac{1}{4}} = \frac{\frac{3}{8}}{\frac{3}{4}} = \frac{3}{8}\cdot\frac{4}{3} = \frac{1}{2}.$$

Now, note that $\displaystyle\lim_{n\to\infty} \frac{n^2+2}{2n^2} = \frac{1}{2} \neq 0$. By the ***n*th term test for divergence,** the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2}{2n^2}$ is **divergent** (so it cannot equal $\frac{1}{2}$).

Finally, using the "known" Taylor series for $\cos(x)$, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\pi}{3}\right)^{2n} = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}.$$

d. [2 points] Consider the curve traced out by the parametric equations:

$$\big(x(t),\ y(t)\big) = \big(t^2,\ \sin(\pi t)\big) \qquad \text{for } t \geq 0.$$

Which, if any, of the following is the **slope** of the tangent line to this curve at $t = 1$?

$\frac{2}{\pi}$ $\quad$ $-\frac{2}{\pi}$ $\quad$ $\frac{\pi}{2}$ $\quad$ $\boxed{-\frac{\pi}{2}}$ $\quad$ 2π $\quad$ -2π $\quad$ NONE

Solution: The slope is given by $\dfrac{dy/dt}{dx/dt}$ at $t = 1$. We have $x'(t) = 2t$ and $y'(t) = \pi\cos(\pi t)$, so $\left.\dfrac{dy/dt}{dx/dt}\right|_{t=1} = \dfrac{y'(1)}{x'(1)} = -\dfrac{\pi}{2}$.

e. [2 points] Which, if any, of the following points given in **polar coordinates** $(\boldsymbol{r}, \boldsymbol{\theta})$ represent the same point as $(x, y) = (-1, 0)$ in the xy-plane?

$\boxed{(r,\theta) = (1, 3\pi)}$ $\quad$ $\boxed{(r,\theta) = (1, \pi)}$ $\quad$ $(r,\theta) = (-1, \pi)$ $\quad$ $\boxed{(r,\theta) = (-1, 0)}$ $\quad$ NONE

Solution: Use the formulas $x = r\cos(\theta)$ and $y = r\sin(\theta)$ to convert each listed point from polar coordinates to Cartesian coordinates. All of them are converted to the point $(x, y) = (-1, 0)$, except for the point $(r, \theta) = (-1, \pi)$, which is converted to the point $(x, y) = (1, 0)$.

"Known" Taylor Series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \leq 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$

Select Values of Trigonometric Functions:

θ	$\sin(\theta)$	$\cos(\theta)$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$