Math 116 — First Midterm — February 10, 2025

Write your 8-digit UMID number very clearly in the box to the right, and fill out the information on the lines below.

Your Initials Only: ______ Your 8-digit UMID number (not uniqname): ____________________

Instructor Name: ______________________________________ Section #: ____________

1. Please neatly write your 8-digit UMID number, your initials, your instructor's first and/or last name, and your section number in the spaces provided.
2. This exam has 11 pages including this cover.
3. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form.* Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	11	
2	15	
3	11	
4	9	
5	9	

Problem	Points	Score
6	5	
7	7	
8	9	
9	12	
10	12	
Total	100	

1. [11 points] Caroline is a water engineer who is monitoring the volume of water in a reservoir over a 16 hour period. The function $r(t)$ gives the rate, in gallons per hour, that water is **flowing into** the reservoir t hours after Caroline begins her measurements. Caroline measures this rate every 2 hours and finds the following values:

t	0	2	4	6	8	10	12	14	16
$r(t)$	100	120	170	230	280	210	190	160	140

a. [2 points] Write an expression involving an integral for the average value of $r(t)$ during the 16 hour period.

Answer: ______________________________

b. [3 points] Find the MID(4) approximation to

$$\int_0^{16} r(t)\, dt.$$

Write out all the terms in your sum. You do not need to simplify, but your final answer should not involve the letter r.

Answer: ______________________________

c. [3 points] Find the RIGHT(4) approximation to

$$\int_0^{16} t\, r(t)\, dt.$$

Note that this is a different integral than the one in part **b.**. Write out all the terms in your sum. You do not need to simplify, but your final answer should not involve the letter r.

Answer: ______________________________

d. [3 points] During the 16 hour period, water is **released from** the reservoir at a constant rate of 200 gallons per hour. At the end of the 16 hours, Caroline finds that the volume of water in the reservoir is $10,000$ gallons. Find an expression involving an integral for the volume of water, in gallons, in the reservoir at the start of the 16 hour period. You do not need to simplify your answer.

Answer: ______________________________

2. [15 points] Let $f(x)$ be an **odd**, **twice-differentiable** function defined for all real numbers. Some values of $f(x)$ and $f'(x)$ are given in the table below:

x	1	2	3	4	5
$f(x)$	3	8	5	0	7
$f'(x)$	2	0	-1	-2	6

Compute the exact value of the following quantities. If there is not enough information provided to answer the question, write “NEI” and clearly indicate why. Show all of your work.

a. [5 points] $\displaystyle\int_{e^{-2}}^{e} \frac{f'(1+\ln x)}{x}\,\mathrm{d}x.$

Answer: ____________________

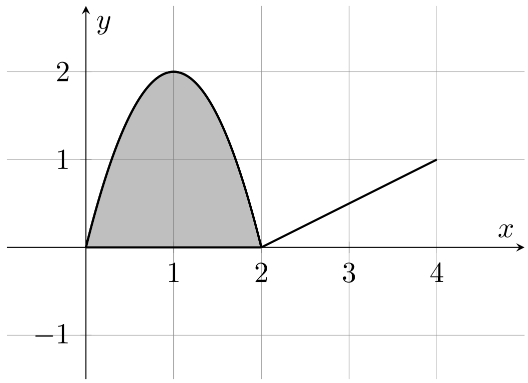
b. [5 points] $\displaystyle\int_{1}^{4} f(x)e^{f(x)}f'(x)\,\mathrm{d}x.$

Answer: ____________________

c. [5 points] $\displaystyle\int_{1}^{9} f''(\sqrt{x})\,\mathrm{d}x.$

Answer: ____________________

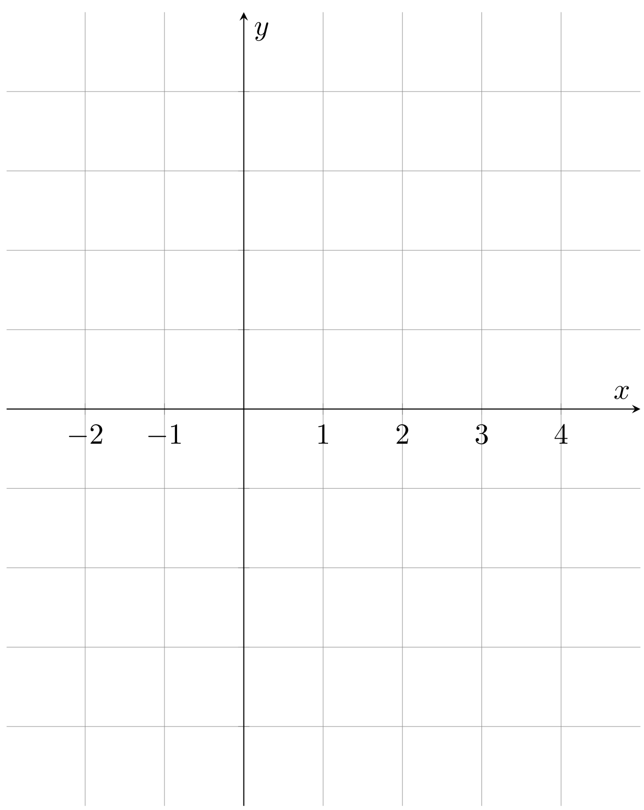
3. [11 points] An **even** function $h(x)$, which is defined for all real numbers, is graphed on the interval $[0, 4]$ below. Note that $h(x)$ is linear on the interval $(2, 4)$, and that the shaded region has area 3.



a. [3 points] The function $h(x)$ has a continuous antiderivative, $H(x)$, which satisfies $H(2) = 2$. Complete the following table of values for $H(x)$.

x	-2	0	2	4
$H(x)$			2	

b. [8 points] Sketch a graph of $H(x)$ on the interval $[-2, 4]$ using the axes provided. Make sure to clearly label the values at the points in the table above and also make it clear where $H(x)$ is increasing or decreasing, and where $H(x)$ is concave up, concave down, or linear.



4. [9 points] Consider the following function:

$$F(x) = 3 + \int_{-1}^{\cos(x)} \frac{e^t}{2+t}\,\mathrm{d}t.$$

a. [2 points] Find a value of a such that $F(a) = 3$. Show your work.

Answer: $a =$ ____________________

b. [3 points] Calculate $F'(x)$.

Answer: $F'(x) =$ ____________________

c. [4 points] Find a function $f(t)$ and constants a and C so that we may rewrite $F(x)$ in the form $\displaystyle\int_a^x f(t)\,\mathrm{d}t + C$. There may be more than one correct answer.

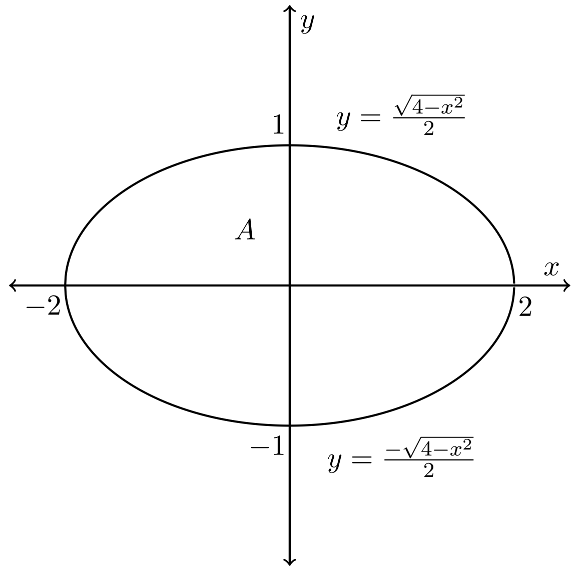
$f(t) =$ ____________________ $a =$ ________ $C =$ ________

5. [9 points]
Mike owns Mike's Sweet Haven, a bakery that specializes in elegant, custom-made baked goods. He uses precise mathematical models to calculate the exact volumes of various bakery items based on their shapes and sizes. This approach ensures he maintains high quality without running out of ingredients or wasting supplies.

He decides to bake artisan bread using region A as the base, which is bounded by the curves

$$y = \frac{\sqrt{4-x^2}}{2}, \text{ and } y = -\frac{\sqrt{4-x^2}}{2}.$$

as illustrated to the right.



a. [5 points] Write an expression involving one or more integrals for the volume of an artisan bread whose base is the region A, and whose cross-sections perpendicular to the x-axis are semicircles. **Do not** evaluate any integrals in your expression.

Answer: ____

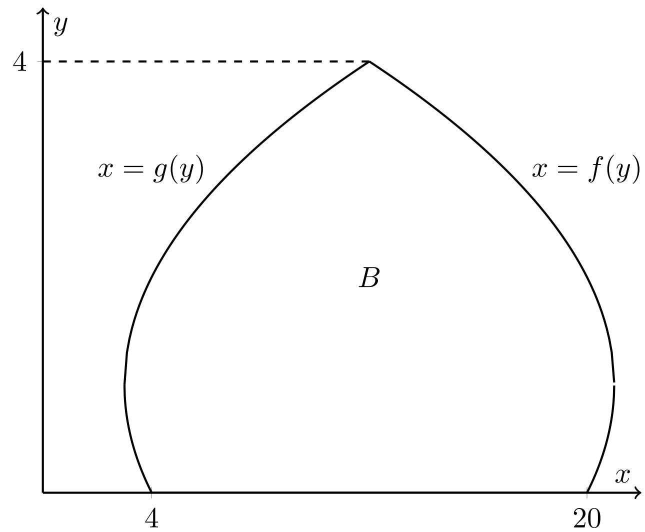
b. [4 points] Determine the perimeter of region A. Write an expression that involves **exactly one** integral.

Answer: ____

6. [5 points]
Mike also plans to make a Bundt cake. He designs the cake using the region B, which is the region in the first quadrant bounded by the curves

$$f(y) = 21 - (y-1)^2,$$
$$g(y) = 3 + (y-1)^2, \text{ and}$$
$$y = 0,$$

as shown below.



Write an expression involving one or more integrals for the volume of the cake generated by revolving the region B about the y-axis. **Do not** evaluate any integrals in your expression. Your final answer should not involve the letters f or g.

Answer: ______________________________

7. [7 points] Use the partial fraction decomposition

$$\frac{x^2 + 11x - 6}{(2 - x)(x^2 + 1)} = \frac{4}{2 - x} + \frac{3x - 5}{x^2 + 1}$$

to evaluate the following indefinite integral, showing all of your work.

$$\int \frac{x^2 + 11x - 6}{(2 - x)(x^2 + 1)}\, \mathrm{d}x$$

Answer: ______________________________

8. [9 points] The following parts are unrelated. No justification is required for your answers.

a. [3 points] Suppose f and g are twice differentiable functions satisfying $f(0) = 0$, $f(1) = 1$, $g(0) = 1$, and $g(1) = 0$. Which of the following **must** be true? Circle **all** correct answers.

i. $\displaystyle\int_0^1 (f(x) - g(x))\,\mathrm{d}x = 0$

ii. $\displaystyle\int_0^1 \big(f'(x) + g'(x)\big)\,\mathrm{d}x = 0$

iii. $\displaystyle\int_0^1 \big(f''(x) - g''(x)\big)\,\mathrm{d}x = 0$

iv. $\displaystyle\int_0^1 x f''(x)\,\mathrm{d}x + 1 = f'(0) + \int_0^1 f''(x)\,\mathrm{d}x$

v. $\displaystyle\int_0^1 e^x g(x)\,\mathrm{d}x = -\int_0^1 e^x g'(x)\,\mathrm{d}x$

vi. NONE OF THESE

b. [3 points] Which of the following are antiderivatives to the function $h(x) = \cos(x^2)$? Circle **all** correct answers.

i. $\dfrac{\sin(x^2)}{2x}$

ii. $\displaystyle\int_0^1 \cos(x^2)\,\mathrm{d}x$

iii. $\displaystyle\int_0^{x^2} \cos(t)\,\mathrm{d}t$

iv. $\displaystyle\int_2^x \cos(t^2)\,\mathrm{d}t$

v. $\displaystyle\int_0^1 \cos(t^2)\,\mathrm{d}t + \int_0^x \cos(t^2)\,\mathrm{d}t$

vi. NONE OF THESE

c. [3 points] For which of the following integrals could the sum

$$\sum_{n=0}^{3} \frac{1}{2}\cos(n)$$

serve as a <u>left Riemann sum</u> approximation? Circle **all** correct answers.

i. $\displaystyle\int_0^3 \frac{1}{2}\cos(x)\,\mathrm{d}x$

ii. $\displaystyle\int_0^4 \frac{1}{2}\cos(x)\,\mathrm{d}x$

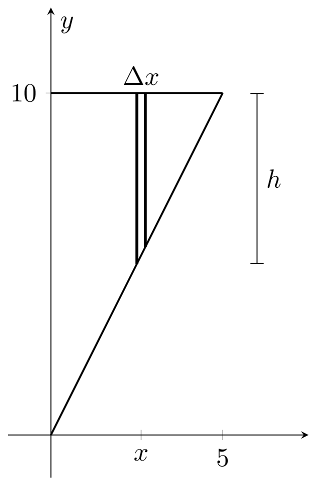
iii. $\displaystyle\int_{-1}^3 \frac{1}{2}\cos(x)\,\mathrm{d}x$

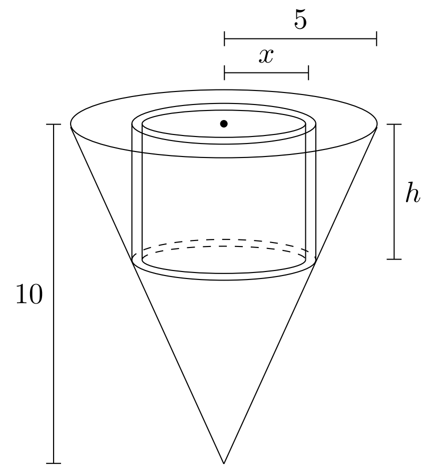
iv. $\displaystyle\int_0^2 \cos(2x)\,\mathrm{d}x$

v. $\displaystyle\int_0^3 \cos(2x)\,\mathrm{d}x$

vi. NONE OF THESE

9. [12 points] Eren is an ice cream vendor who loves to experiment with new ideas. He decides to create an ice cream treat by rotating the region bounded by the y-axis, $y = 2x$, and $y = 10$ about the y-axis, as shown in the figure below, where all distances are measured in centimeters. The density of the ice cream at a point x centimeters from the y-axis is given by $\delta(x) = \sqrt{x^2+1}$ grams per cubic centimeter (g/cm^3).





a. [2 points] Consider the thin vertical strip of the region depicted above on the left, which is located x centimeters from the y-axis, and has height h and small thickness Δx. Find a formula for h in terms of x.

Answer: $h =$ ______________________

b. [4 points] When the strip above is rotated around the y-axis, it forms a thin **cylindrical shell** (depicted above on the right). Write an expression which approximates the **volume** of that shell. Your answer should not involve the letter h. **Include units.**

Answer: ______________________ **Units:** __________

c. [3 points] Write an expression that approximates the **mass** of the thin cylindrical shell of ice cream described in part **b.** Your answer should not involve the letters h or δ. **Include units.**

Answer: ______________________ **Units:** __________

d. [3 points] Write an expression involving one or more integrals that represents the **total mass** of the ice cream in the treat. **Do not** evaluate any integrals in your expression. Your answer should not involve the letters h or δ. **Include units.**

Answer: ______________________ **Units:** __________

10. [12 points] For the following questions, determine if the statement is ALWAYS true, SOMETIMES true, or NEVER true, and circle the corresponding answer. Justification is not required.

a. [2 points] If $a(x)$ is a concave down differentiable function, and MID(20) and TRAP(20) estimate $\int_{-1}^{1} a(x)\,dx$, then

$$\text{MID}(20) < \text{TRAP}(20).$$

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

b. [2 points] If $b(x)$ is an increasing, concave up differentiable function, and LEFT(12) and MID(12) estimate $\int_{-1}^{1} b(x)\,dx$, then

$$\text{LEFT}(12) \leq \text{MID}(12) \leq \int_{-1}^{1} b(x)\,dx.$$

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

c. [2 points] Suppose that $f(x)$ is an increasing differentiable function, and that LEFT(2) and LEFT(4) both estimate $\int_{-1}^{1} f(x)\,dx$. Then

$$\text{LEFT}(2) \leq \text{LEFT}(4) \leq \int_{-1}^{1} f(x)\,dx.$$

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

d. [2 points] Suppose that $g(x)$ is a differentiable function which is decreasing and concave up. Let $G(x) = \int_{0}^{x} g(t)\,dt$, and suppose that LEFT(10) estimates $\int_{-1}^{1} G(x)\,dx$. Then LEFT(10) gives an overestimate.

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

e. [2 points] Suppose that $g(x)$ is a differentiable function which is decreasing and concave up. Let $G(x) = \int_{0}^{x} g(t)\,dt$, and suppose that MID(10) estimates $\int_{-1}^{1} G(x)\,dx$. Then MID(10) gives an overestimate.

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

f. [2 points] Suppose that a thin circular plate has radius 3 centimeters, and that the density of the plate, in grams per square centimeter, at a radial distance r centimeters from the center is given by the function $p(r)$. Suppose also that $p(r)$ is an increasing function. Then the total mass of the plate is no more than

$$2\pi\left(p(1) + 2p(2) + 3p(3)\right).$$

Circle one: **ALWAYS** **SOMETIMES** **NEVER**