Math 116 — Final Exam — April 25, 2025

Write your 8-digit UMID number very clearly in the box to the right, and fill out the information on the lines below.

Your Initials Only: ______ Your 8-digit UMID number (not uniqname): ________________

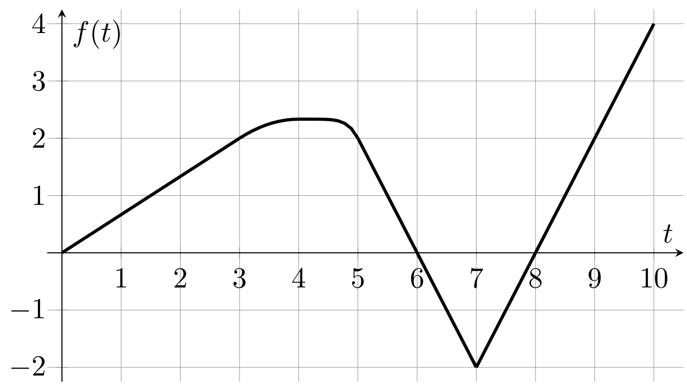
Instructor Name: ____________________________ Section #: ________

1. Please neatly write your 8-digit UMID number, your initials, your instructor's first and/or last name, and your section number in the spaces provided.
2. This exam has 12 pages including this cover.
3. There are 11 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form.* Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	13	
2	6	
3	8	
4	6	
5	12	
6	10	

Problem	Points	Score
7	10	
8	6	
9	12	
10	9	
11	8	
Total	100	

1. [13 points] Caroline uses a remote-controlled boat to survey a reservoir. The boat starts at the point $(x, y) = (0, 0)$, and after t seconds is positioned at $x = f(t)$ and $y = g(t)$. A graph of $f(t)$ and a formula for $g(t)$ are given below. Note that $f(t)$ is linear on the the intervals $[0, 3]$, $[5, 7]$, and $[7, 10]$, and has a local maximum at $t = 4$.



$$g(t) = 12\cos\left(\frac{\pi}{2}t\right) - 12$$

For each of the following parts, your final answer should **not** include the letters f or g.

a. [2 points] Where is the boat located after 10 seconds?

Answer: $x =$ ____________ and $y =$ ____________

b. [3 points] Are there any times during these 10 seconds at which the boat comes to a complete stop? If so, list all such times. If not, write NONE.

Answer: $t =$ ____________

c. [4 points] Write an expression involving one or more integrals for the total distance traveled by the boat during the **first 3 seconds**. Do not evaluate any integrals in your answer.

Answer: ____________

d. [4 points] What is the tangent line to the boat's path at $t = 9$? Give your answer in cartesian form.

Answer: $y =$ ____________

2. [6 points] Compute the **exact** value of each of the following, if possible. Your answers should not involve integration signs, ellipses or sigma notation. For any values which do not exist, write **DNE**. You do not need to show work.

a. [2 points] The value of $G'(2)$ if $G(x) = \int_1^{3-x} e^{t^3}\,dt$.

Answer: ____________________

b. [2 points] The infinite sum $-1 + \frac{5^2}{2!} - \frac{5^4}{4!} + \frac{5^6}{6!} - \cdots + \frac{(-1)^{n+1}5^{2n}}{(2n)!} + \cdots$.

Answer: ____________________

c. [2 points] The infinite sum $\sum_{n=0}^{\infty} 3(4^n)$.

Answer: ____________________

3. [8 points] The two parts of this problem ask about **the same** series. No justification is required for your answers.

a. [4 points] Which of the following series converge? Circle **all** options that apply.

i. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$

ii. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+3}$

iii. $\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^{1/2}}$

iv. $\sum_{n=1}^{\infty} \frac{(-4)^n}{5^n}$

v. $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

vi. $\sum_{n=3}^{\infty} \frac{(-1)^n}{n\ln(n)}$

vii. NONE OF THESE

b. [4 points] Which of the following series converge **conditionally**? Circle **all** options that apply.

i. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$

ii. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+3}$

iii. $\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^{1/2}}$

iv. $\sum_{n=1}^{\infty} \frac{(-4)^n}{5^n}$

v. $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

vi. $\sum_{n=3}^{\infty} \frac{(-1)^n}{n\ln(n)}$

vii. NONE OF THESE

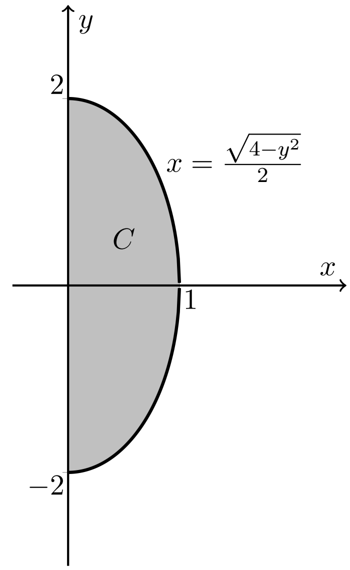
4. [6 points]

Mike owns Mike's Sweet Haven, a bakery that specializes in elegant, custom-made baked goods. With summer approaching, he decides to try something new.

He plans to create a new type of chocolate using the shaded region C, which is bounded by the curves

$$x = \frac{\sqrt{4-y^2}}{2}, \text{ and } x = 0.$$

as illustrated to the right.



a. [4 points] Write an expression involving one or more integrals for the volume of the chocolate obtained by revolving the region C about the y-axis. **Do not** evaluate any integrals in your expression.

Answer: ______________________________

b. [2 points] If Mike were to take a TRAP(4) estimate of the integral you obtained in part **a.**, would he get an underestimate or an overestimate of the volume of the chocolate? No justification is required.

Circle one: **Underestimate** **Overestimate**

5. [12 points] The Taylor series centered at $x = -1$ for a function $f(x)$ is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{9^n (n!)^2}{(2n+1)!}(x+1)^{2n+1}$$

a. [7 points] Determine the **radius** of convergence of the Taylor series above. Show all of your work. You do **not** need to find the interval of convergence.

Answer: ______________________

b. [5 points] Find $f^{(2025)}(-1)$ and $f^{(2026)}(-1)$. You do not need to simplify your answers.

Answer: $f^{(2025)}(-1) =$ ______________________ and $f^{(2026)}(-1) =$ ______________________

6. [10 points] A power series centered at $x = 3$ is given by

$$\sum_{n=1}^{\infty} \frac{2n+1}{5^n(n^2+1)}(x-3)^n.$$

The radius of convergence of this power series is 5 (do **not** show this). Find the **interval** of convergence of this power series. Show all your work, including full justification for series behavior.

Interval of convergence: ______________________

7. [10 points] Consider the function

$$g(x) = \frac{1}{3}\cos(x^2) - x\sin(x).$$

a. [5 points] Give the first three non-zero terms of the Taylor series of $g(x)$ centered about $x = 0$. Show all your work.

Answer: ______________________

b. [5 points] The function $g(x)$ has a continuous antiderivative, $G(x)$, with a Taylor series that converges for all x. Given that $G(0) = 8$, find the first four non-zero terms of the Taylor series for $G(x)$ centered about $x = 0$. Show all your work.

Answer: ______________________

8. [6 points] A team of miners is working to extract a box of minerals from a deep pit. The box weighs 40 lbs, and the rope used to lift it weighs 3 lbs per foot. Initially, when the box is at the bottom of the pit, the rope is 60 feet long. As the box is lifted, the miners do not need to lift the portion of the rope that has already been "reeled in", that is, the part that has reached the top of the pit.

a. [3 points] At a certain moment, the box has already been lifted h feet above the ground. Find an expression for the total weight, in pounds, of the box together with the attached rope that has not yet been reeled in.

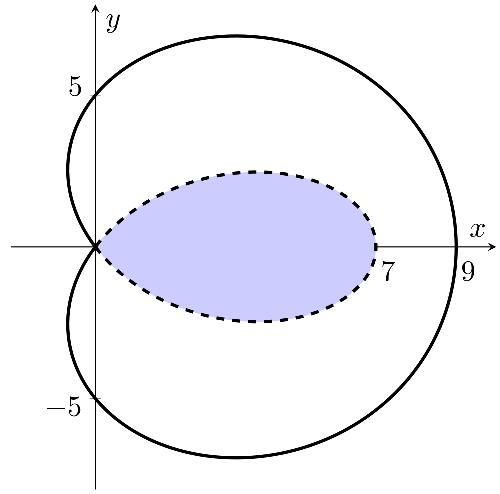
Answer: ______________________________

b. [3 points] Using your expression from part (a), find an expression involving one or more integrals that represents the total work done on the box and attached rope, in foot-pounds, to lift the box from the base of the pit to a point 35 feet above its original position. Do not evaluate any integrals that appear in your answer.

Answer: ______________________________

9. [12 points]

Elena, a talented landscape architect, envisions a park whose shape is defined by the polar curve $r = 5 + 8\cos(\theta) - 4\cos^2(\theta)$, as illustrated to the right. In her design, the inner loop of the curve serves as an ideal location for a lake, represented by the shaded region. The solid outer curve in the diagram represents the walking trail that winds around the park.



a. [4 points] Using the factorization $5 + 8\cos(\theta) - 4\cos^2(\theta) = (1 + 2\cos(\theta))(5 - 2\cos(\theta))$, find the values of θ in the interval $[0, 2\pi)$ for which the curve passes through the origin.

Answer: $\theta =$ ______________________

b. [4 points] To determine the amount of water required to fill the lake, Elena wants to calculate the area of the surface of the lake. Write an expression involving one or more integrals that represents the area of the shaded region. Do not evaluate the integral(s).

Answer: ______________________

c. [4 points] Recall that the solid outer curve in the diagram represents the walking trail. Write an expression involving one or more integrals that represents the total length of the walking trail. Do not evaluate the integral(s).

Answer: ______________________

10. [9 points] Let a be a real number. Consider the following integral

$$\int_0^1 ax\ln(x)\,\mathrm{d}x$$

a. [8 points] Show that the above integral converges by using a **direct computation** and find its value in terms of a. Be sure to show your full computation, and be sure to use **proper notation**.

Answer: $\displaystyle\int_0^1 ax\ln(x)\,\mathrm{d}x =$ ________________________________

b. [1 point] Find the value of a so that the function

$$p(x) = \begin{cases} ax\ln(x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function (pdf).

Answer: $a =$ ________________________

11. [8 points] The parts of this problem are unrelated. No justification is required for your answers.

a. [2 points] Let $p(x)$ be a probability density function (pdf). Then, which of the following functions must also be probability density function? Circle **all** options which apply.

i. $2p(x)$

ii. $p(2x)$

iii. $2\,p(2x)$

iv. $2x\,p(x^2)$

v. $3x^2\,p(x^3)$

vi. NONE OF THESE

b. [2 points] The cartesian coordinates of a point A are $(x, y) = \left(\sqrt{2}, -\sqrt{2}\right)$. Which of the following represent the polar coordinates of point A? Circle **all** options which apply.

i. $(r, \theta) = \left(\sqrt{2}, -\frac{\pi}{4}\right)$

ii. $(r, \theta) = \left(2, -\frac{\pi}{4}\right)$

iii. $(r, \theta) = \left(2, \frac{7\pi}{4}\right)$

iv. $(r, \theta) = \left(-2, \frac{\pi}{4}\right)$

v. $(r, \theta) = \left(-2, \frac{3\pi}{4}\right)$

vi. NONE OF THESE

c. [2 points] A power series $\sum_{n=0}^{\infty} C_n(x-a)^n$ converges at $x = -4$ and diverges at $x = 2$. Which of the following values could be the center, a, of the power series? Circle **all** options which apply.

i. $a = -2$

ii. $a = -1$

iii. $a = 0$

iv. $a = 1$

v. $a = 2$

vi. NONE OF THESE

d. [2 points] At which of the following values of θ in $[0, \pi)$ does the curve $r = \cos(\theta)$ have a horizontal tangent line? Circle **all** options which apply.

i. $\theta = 0$

ii. $\theta = \frac{\pi}{4}$

iii. $\theta = \frac{\pi}{3}$

iv. $\theta = \frac{\pi}{2}$

v. $\theta = \frac{3\pi}{4}$

vi. NONE OF THESE

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \le 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$

Select Values of Trigonometric Functions:

θ	$\sin\theta$	$\cos\theta$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$