Math 116 — Second Midterm — March 24, 2025

EXAM SOLUTIONS

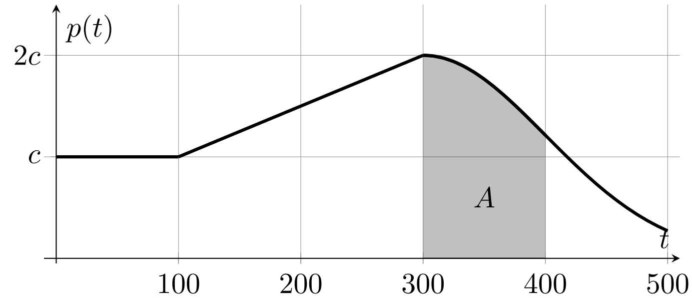
1. Please neatly write your 8-digit UMID number, your initials, your instructor's first and/or last name, and your section number in the spaces provided.
2. This exam has 14 pages including this cover.
3. There are 11 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	9	
2	4	
3	7	
4	7	
5	6	
6	12	

Problem	Points	Score
7	7	
8	12	
9	12	
10	12	
11	12	
Total	100	

1. [9 points] Sunita is a horticulturist, who has been cultivating tomato plants. Let $p(t)$ be the probability denisty function (pdf) for the lifespan t of a tomato plant in her facility, measured in days.

A **partial** graph of $p(t)$ is shown below. Note that $p(t)$ is piecewise linear on the interval $[0, 300]$ and that $p(t) = 0$ for all $t < 0$. The value of the shaded area between $p(t)$ and the t-axis on $[300, 400]$ is given by the positive number A.



a. [2 points] Suppose that $p(800) = 0.0002$. Complete the following sentence:

"The probability that a tomato plant has a lifespan between 775 and 825 days is . . . "

Solution: . . . approximately $(0.0002)(50) = 0.01 = 1\%$."

b. [2 points] The median lifespan of a tomato plant is 300 days. Find the value of c.

Solution: Since the median for the pdf $p(t)$ is 300, we must have $\int_0^{300} p(t)\,dt = 0.5$. Computing the area under the graph of $p(t)$ from 0 to 300, we obtain that $400c = 0.5$. Therefore, $c = \frac{1}{800}$.

Answer: $c =$ $\frac{1}{800}$

c. [2 points] Additionally, suppose that there is a 80% chance that a tomato plant in the facility has a lifespan of 400 days or fewer. Find the value of A.

Solution: From part **b**, we know that there is a 50% chance that a tomato in the facility has a lifespan of 300 days or fewer. Combining this with the information given in this part, we find that there is a 30% chance that a tomato plant in the facility has a lifespan between 300 and 400 days. Therefore, we must have $A = 0.3$.

Answer: $A =$ 0.3

d. [3 points] Let $S(t)$ be the function which gives the probability that a tomato plant has a lifespan of t or **more** days in Sunita's facility. Which of the following statements **must** be true, given the assumptions from parts **b.** and **c.**?

i. $\boxed{S(350) < 0.5}$

ii. $S(350) < 0.2$

iii. $\boxed{\lim_{t\to\infty} S(t) = 0}$

iv. $\lim_{t\to\infty} S(t) = 1$

v. $S(600) \leq S(700)$

vi. $\boxed{S(600) \geq S(700)}$

vii. NONE OF THESE

2. [4 points] Compute the following limit. Fully justify your answer including using **proper limit notation**.

$$\lim_{x\to 1} \frac{\sin(\ln(x))}{x^2-1}$$

Solution: Since $\ln 1 = 0$, the limit is in the indeterminate form "$\frac{0}{0}$". We have

$$\lim_{x\to 1} \frac{\sin(\ln(x))}{x^2-1} \overset{\text{L'H}}{=} \lim_{x\to 1} \frac{\frac{\cos(\ln(x))}{x}}{2x} = \frac{\cos(\ln(1))}{2(1)^2} = \frac{\cos(0)}{2} = \frac{1}{2}.$$

Answer: $\lim_{x\to 1} \frac{\sin(\ln(x))}{x^2-1} =$ $\frac{1}{2}$

3. [7 points] **Compute** the value of the following improper integral if it converges.
If it does not converge, use a **direct computation** of the integral to show its divergence.
Be sure to show your full computation, and be sure to use **proper notation**.

$$\int_0^\infty \frac{x}{e^{3x^2}}\,\mathrm{d}x$$

Solution: By definition,

$$\int_0^\infty \frac{x}{e^{3x^2}}\,\mathrm{d}x = \lim_{b\to\infty} \int_0^b \frac{x}{e^{3x^2}}\,\mathrm{d}x.$$

Set $u = 3x^2$. Then $\mathrm{d}u = 6x\,\mathrm{d}x$. Using the method of substitution,

$$\int \frac{x}{e^{3x^2}}\,\mathrm{d}x = \frac{1}{6}\int e^{-u}\,\mathrm{d}u = -\frac{e^{-u}}{6} + C = -\frac{e^{-3x^2}}{6} + C.$$

Therefore,

$$\begin{aligned}
\int_0^\infty \frac{x}{e^{3x^2}}\,\mathrm{d}x &= \lim_{b\to\infty} \int_0^b \frac{x}{e^{3x^2}}\,\mathrm{d}x \\
&= \lim_{b\to\infty} \left[-\frac{e^{-3x^2}}{6}\Bigg|_0^b \right] \\
&= \lim_{b\to\infty} \left[-\frac{e^{-3b^2}}{6} + \frac{1}{6} \right] \\
&= 0 + \frac{1}{6} \\
&= \frac{1}{6}.
\end{aligned}$$

Circle one: **Diverges** $\boxed{\textbf{Converges to } \frac{1}{6}}$

4. [7 points] Determine if the following series converges or diverges using the **Limit Comparison Test**, and circle the corresponding word. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests you use.

$$\sum_{n=1}^{\infty} \frac{2n-1}{\sqrt{9n^4-10n+3}}$$

Circle one: **Converges** $\boxed{\textbf{Diverges}}$

*Justification (using the **Limit Comparison Test**):*

Solution: We compare the given series with the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. We have

$$\lim_{n\to\infty} \frac{\frac{2n-1}{\sqrt{9n^4-10n+3}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{(2n-1)n}{\sqrt{9n^4-10n+3}} = \lim_{n\to\infty} \frac{2n^2}{3n^2} = \frac{2}{3} > 0.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-test ($p=1$). Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{2n-1}{\sqrt{9n^4-10n+3}}$ diverges by the Limit Comparison Test (LCT).

5. [6 points] Determine if the following series converges or diverges, and circle the corresponding word. **Fully justify** your answer including using **proper notation** and **showing mechanics** of any tests you use.

$$\sum_{n=1}^{\infty} \frac{4^n}{n!}$$

Circle one: **Converges** **Diverges**

Justification:

Solution: We use the Ratio Test to determine whether the given series converges or diverges. First, we form

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{4^{n+1}}{(n+1)!}}{\frac{4^n}{n!}} = \frac{4^{n+1}}{4^n} \cdot \frac{n!}{(n+1)!} = \frac{4}{n+1}.$$

Thus

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{4}{n+1} = 0 < 1.$$

Hence, the series $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{n!}$ converges by the Ratio test.

6. [12 points] Kalani and his friends are in Honolulu, Hawaii, enjoying their spring break. One day, they decide to try a new virtual surfing game. Let $P(t)$ denote the cumulative distribution function (cdf) representing the probability that a first-time player takes t minutes or less to complete the game. According to the developer's data, the formula of $P(t)$ is given by

$$P(t) = \begin{cases} 0, & t \leq 0, \\ \dfrac{a}{9}t^2, & 0 < t \leq 3, \\ \dfrac{b}{2} - ae^{3-t}, & t > 3. \end{cases}$$

where $a > 0$ and $b > 0$ are constants.

a. [4 points] The function $P(t)$ is a **continuous** cumulative distribution function (cdf). Find the values of a and b.

Solution: Since $P(t)$ is a cumulative distribution function (cdf), it must satisfy $\lim\limits_{t\to\infty} P(t) = 1$. That is,

$$\lim_{t\to\infty}\left(\frac{b}{2} - ae^{3-t}\right) = \frac{b}{2} - 0 = \frac{b}{2} = 1$$

Therefore, $b = 2$. Since $P(t)$ is a continuous function, the following must hold:

$$\frac{a}{9}\cdot 9 = \frac{2}{2} - ae^{3-3} \implies a = 1 - a$$

Hence, $a = \dfrac{1}{2}$.

Answer: $a =$ $\dfrac{1}{2}$ and $b =$ 2

b. [2 points] Write an expression for the probability that a first-time player takes at least 1 minute and at most 7 minutes to complete the game. Your answer may include the letters a and b, but it should not involve the letter P. Your answer should **not** include integrals.

Solution: The probability that a first-time player takes at least 1 minute and at most 7 minutes to complete the game is given by

$$P(7) - P(1) = \left(\frac{b}{2} - ae^{3-7}\right) - \frac{a(1)^2}{9} = \left(\frac{b}{2} - ae^{-4}\right) - \frac{a}{9} = \left(1 - \frac{e^{-4}}{2}\right) - \frac{1}{18}.$$

Answer: $\dfrac{17}{18} - \dfrac{e^{-4}}{2}$

c. [3 points] Write a piecewise-defined formula for $p(t)$, the probability density function (pdf) corresponding to $P(t)$. Your answer may include the letters a and b, but it should not include the letter P.

Solution: Since $P(t)$ is an anti-derivative of $p(t)$, we have $p(t) = P'(t)$. Therefore,

$$p(t) = \begin{cases} 0, & t \le 0, \\ \dfrac{t}{9}, & 0 < t \le 3, \\ \dfrac{e^{3-t}}{2}, & t > 3. \end{cases}$$

d. [3 points] Write an expression involving one or more integrals that represents the mean time (in minutes) it takes for a first-time player to complete the virtual surfing game. Your answer may include the letters a and b, but it should not involve the letters P or p. **Do not evaluate your integral(s).**

Solution: The mean time is given by

$$\int_0^3 \frac{t^2}{9}\,\mathrm{d}t + \int_3^\infty \frac{te^{3-t}}{2}\,\mathrm{d}t$$

Answer: $\int_0^3 \frac{t^2}{9}\,\mathbf{d}t + \int_3^\infty \frac{te^{3-t}}{2}\,\mathbf{d}t$

7. [7 points] Determine whether the following improper integral converges or diverges and circle the corresponding word. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests you use. You do not need to calculate the value of the integral if it converges.

$$\int_1^\infty \frac{\cos\left(\frac{1}{t}\right)}{t^{2/5}}\, \mathrm{d}t$$

Circle one: **Converges** $\boxed{\textbf{Diverges}}$

Justification:

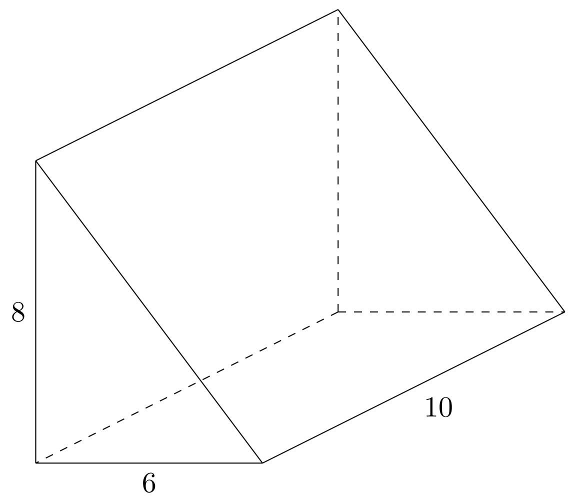
Solution: We use the Direct Comparison Test (DCT) to determine the convergence or divergence of the given integral. Observe that $\cos\left(\dfrac{1}{t}\right) \geq \cos(1)$ for $t \geq 1$. Therefore,

$$\frac{\cos\left(\frac{1}{t}\right)}{t^{2/5}} \geq \frac{\cos(1)}{t^{2/5}}, \quad t \geq 1.$$

The integral $\displaystyle\int_1^\infty \frac{\cos(1)}{t^{2/5}}\, \mathrm{d}t$ diverges by the p-test ($p = 2/5$). Therefore, $\displaystyle\int_1^\infty \frac{\cos\left(\frac{1}{t}\right)}{t^{2/5}}\, \mathrm{d}t$ diverges by the Direct Comparison Test (DCT).

8. [12 points]

Caroline the water engineer is helping to clean out a large tank which is partially filled with sludge. The tank is in the shape of a right triangular prism, with height 8 meters, width 6 meters, and length 10 meters, as depicted to the right. The sludge fills the tank to a depth of 5 meters, so that the top of the sludge is 3 meters below the top of the tank. You may assume that the acceleration due to gravity is $g = 9.8\text{m/s}^2$.



a. [2 points] The width of a horizontal slice of sludge at the bottom of the tank is 6 meters. Find an expression for the width, in meters, of a horizontal slice which is h meters above the bottom of the tank.

Solution: Denote by w the width of a horizontal slice of sludge located h meters above the bottom of the tank. We use similar triangles to determine the relationship between w and h:

$$\frac{w}{6} = \frac{8-h}{8} \implies w = \frac{6(8-h)}{8} = \frac{3(8-h)}{4}.$$

Answer: $\frac{3(8-h)}{4}$

b. [5 points] The density, in kilograms per cubic meter, of the sludge at a height h meters above the bottom of the tank is given by the function $p(h)$. Find an expression which approximates the **weight** of a horizontal slice of the sludge which is h meters above the bottom of the tank, and which has small thickness Δh. Your expression should not contain any integrals. **Include units**.

Solution: The approximate volume ΔV of a horizontal slice of sludge at a height h meters above the bottom of the tank, with a small thickness Δh, is given by

$$\Delta V \approx 10w\Delta h = 10\left(\frac{3(8-h)}{4}\right)\Delta h = \frac{15(8-h)}{2}\Delta h.$$

Thus, the approximate mass Δm of this slice is

$$\Delta m \approx \Delta V \times p(h) = \frac{15(8-h)p(h)}{2}\Delta h$$

And, therefore, the approximate weight ΔF of this slice is

$$\Delta F \approx \Delta m \times g = \frac{15g(8-h)p(h)}{2}\Delta h$$

Answer: $\frac{15g(8-h)p(h)}{2}\Delta h$ **Units:** Newton (N) or kgms^{-2}

c. [5 points] Recall that the sludge fills the tank to a depth of 5 meters, so that the top of the sludge is 3 meters below the top of the tank. Write an expression involving one or more integrals that gives the total amount of work needed to pump all the sludge to the top of the tank. Do not evaluate your integral(s). **Include units.**

Solution: Denote by ΔW the approximate work required to pump a horizontal slice of sludge at a height h meters above the bottom of the tank, with a small thickness Δh. This slice is moved a distance of $(8-h)$ meters to be pumped out. We have

$$\Delta W \approx \Delta F \times (8-h) = \frac{15g(8-h)^2 p(h)}{2}\Delta h$$

Since the sludge fills the tank to a depth of 5 meters, the total work required to pump all the sludge to the top of the tank is given by

$$\int_0^5 \frac{15g(8-h)^2 p(h)}{2}\,\mathrm{d}h$$

Answer: $\int_0^5 \frac{15g(8-h)^2 p(h)}{2}\,\mathrm{d}h$ **Units:** Joules (J) or kgm^{2}s^{-2}

9. [12 points] For each of the following sequences, defined for integers $n \geq 1$, decide whether the sequence is monotone increasing, monotone decreasing, or not monotone, and whether it is bounded or unbounded. Circle your answers. No justification is required.

(i) $a_n = (-1)^n \left(1 + \frac{1}{n}\right)$

*Circle **all** that apply:*

Monotone Increasing **Monotone Decreasing** $\boxed{\textbf{Not Monotone}}$

$\boxed{\textbf{Bounded}}$ **Unbounded**

(ii) $b_n = \dfrac{1}{1 + \ln(n)}$

*Circle **all** that apply:*

Monotone Increasing $\boxed{\textbf{Monotone Decreasing}}$ **Not Monotone**

$\boxed{\textbf{Bounded}}$ **Unbounded**

(iii) $c_n = \displaystyle\sum_{k=1}^{n} \frac{(-3)^k}{5^k}$

*Circle **all** that apply:*

Monotone Increasing **Monotone Decreasing** $\boxed{\textbf{Not Monotone}}$

$\boxed{\textbf{Bounded}}$ **Unbounded**

(iv) $r_n = \displaystyle\sum_{k=1}^{n} \left(1 + \frac{1}{k}\right)$

*Circle **all** that apply:*

$\boxed{\textbf{Monotone Increasing}}$ **Monotone Decreasing** **Not Monotone**

Bounded $\boxed{\textbf{Unbounded}}$

(v) $s_n = \displaystyle\int_{1/2^n}^{1} \frac{1}{\sqrt{x}}\, \mathrm{d}x$

*Circle **all** that apply:*

$\boxed{\textbf{Monotone Increasing}}$ **Monotone Decreasing** **Not Monotone**

$\boxed{\textbf{Bounded}}$ **Unbounded**

(vi) $t_n = \displaystyle\sum_{k=2}^{n+1} \frac{1}{k \ln k}$

*Circle **all** that apply:*

$\boxed{\textbf{Monotone Increasing}}$ **Monotone Decreasing** **Not Monotone**

Bounded $\boxed{\textbf{Unbounded}}$

10. [12 points] In *The Great British Rake-Off*, contestants compete for the grand prize - a packet of assorted seeds and a rake stand. One potential contestant, Shane, prepares by raking some of the leaves from his yard each morning. Initially, there are 100kg of leaves on the yard. Each morning, he removes 80% of the leaves on the yard. Over the course of each day, an additional L kg of leaves settle onto the yard, for some constant L.

a. [5 points] Let S_n denote the total mass of leaves, in kg, in the yard immediately *before* Shane starts to rake on the nth day, so that $S_1 = 100$. Find expressions for the values of S_2, S_3 and S_4. You do not need to simplify your expressions, and they may be given in terms of L.

Solution:

$$\begin{aligned} S_1 &= 100 \\ S_2 &= 0.2S_1 + L = (0.2)(100) + L \\ S_3 &= 0.2S_2 + L = (0.2)^2 100 + (0.2)L + L \\ S_4 &= 0.2S_3 + L = (0.2)^3 100 + (0.2)^2 L + (0.2)L + L \end{aligned}$$

Answer: $S_2 =$ $(0.2)(100) + L$

Answer: $S_3 =$ $(0.2)^2 100 + (0.2)L + L$

Answer: $S_4 =$ $(0.2)^3 100 + (0.2)^2 L + (0.2)L + L$

b. [5 points] Find a closed-form expression for S_n. Closed form means your answer should not include ellipses or sigma notation, and should NOT be recursive. You do not need to simplify your expression and it may be given in terms of L.

Solution:

$$\begin{aligned} S_n &= 0.2^{n-1}(100) + (0.2^{n-2} + \cdots + 0.2 + 1)L \\ &= 0.2^{n-1}(100) + \frac{(1 - 0.2^{n-1})L}{1 - 0.2} \end{aligned}$$

Answer: $S_n =$ $0.2^{n-1}(100) + \dfrac{(1 - 0.2^{n-1})L}{1 - 0.2}$

c. [2 points] If Shane ever goes to rake leaves in the morning and finds that there are less than 20kg of leaves in his yard, he will feel ready to enter *The Great British Rake-Off*. For what values of L will Shane eventually meet his goal?

Solution: Rearranging the expression for S_n, we obtain

$$S_n = \frac{L}{0.8} + \left(100 - \frac{L}{0.8}\right)(0.2)^{n-1}.$$

If $L \geq 80$, then $100 - \frac{L}{0.8} < 0$, implying that S_n is monotone increasing. Therefore, if $L \geq 80$, we have $S_n \geq S_1 = 100$. Thus, if $L \geq 80$, Shane will never meet his goal.

Now assume that $L < 80$. In this case, $100 - \frac{L}{0.8} > 0$, so S_n is monotone decreasing. Shane will eventually meet his goal if

$$\lim_{n\to\infty} S_n < 20.$$

Since $\lim_{n\to\infty}(0.2)^{n-1} = 0$, we must have

$$\frac{L}{0.8} < 20 \implies L < 20(0.8) = 16.$$

Answer: For $L <$ 16

11. [12 points] For the following questions, determine if the statement is ALWAYS true, SOMETIMES true, or NEVER true, and circle the corresponding answer. Justification is not required.

a. [2 points] Let $F(x)$ be a cumulative distribution function (cdf) that is continuous for all x. Then, the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n(1-F(n))$ converges.

Circle one: $\boxed{\textbf{ALWAYS}}$ **SOMETIMES** **NEVER**

b. [2 points] Let $f(x)$ be a non-negative, continuous function for all $x \geq 1$, and suppose that $f(x)$ is decreasing. Then, the integral $\displaystyle\int_1^{\infty}\frac{f(x)}{x}\,dx$ converges.

Circle one: **ALWAYS** $\boxed{\textbf{SOMETIMES}}$ **NEVER**

c. [2 points] Let $g(x)$ be a continuous function and suppose that for all n, $s_n = g(n)$. If $\displaystyle\int_1^{\infty} g(x)\,dx$ diverges, then $\displaystyle\sum_{n=1}^{\infty} s_n$ also diverges.

Circle one: **ALWAYS** $\boxed{\textbf{SOMETIMES}}$ **NEVER**

d. [2 points] Let $h(x)$ be a non-negative, continuous function for all x, and suppose that $h(x)$ is decreasing. Let $a_n = h(n)$ for all n. If $\displaystyle\int_1^{\infty} xh(x)\,dx$ converges, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ also converges.

Circle one: $\boxed{\textbf{ALWAYS}}$ **SOMETIMES** **NEVER**

e. [2 points] Let $b_n \geq 0$ and $c_n \geq 0$ for all n. Suppose that the series $\displaystyle\sum_{n=1}^{\infty} b_n$ converges and that the sequence c_n also converges. Then, the series $\displaystyle\sum_{n=1}^{\infty} b_n 2^{c_n}$ diverges.

Circle one: **ALWAYS** **SOMETIMES** $\boxed{\textbf{NEVER}}$

f. [2 points] Suppose that d_n is a monotonic decreasing sequence of positive numbers that converges to 0. Furthermore, assume that $\displaystyle\lim_{n\to\infty}\frac{d_n}{1/n^2} = 5$. Then, the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n d_n$ is conditionally convergent.

Circle one: **ALWAYS** **SOMETIMES** $\boxed{\textbf{NEVER}}$