Math 116 — Final Exam — April 25, 2025

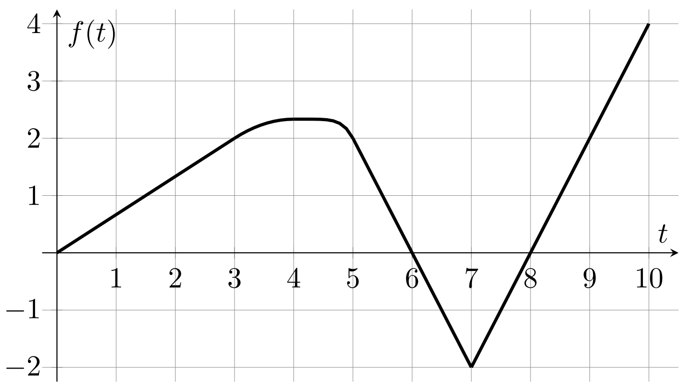
EXAM SOLUTIONS

1. Please neatly write your 8-digit UMID number, your initials, your instructor's first and/or last name, and your section number in the spaces provided.
2. This exam has 14 pages including this cover.
3. There are 11 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	13	
2	6	
3	8	
4	6	
5	12	
6	10	

Problem	Points	Score
7	10	
8	6	
9	12	
10	9	
11	8	
Total	100	

1. [13 points] Caroline uses a remote-controlled boat to survey a reservoir. The boat starts at the point $(x, y) = (0, 0)$, and after t seconds is positioned at $x = f(t)$ and $y = g(t)$. A graph of $f(t)$ and a formula for $g(t)$ are given below. Note that $f(t)$ is linear on the the intervals $[0, 3]$, $[5, 7]$, and $[7, 10]$, and has a local maximum at $t = 4$.



$$g(t) = 12\cos\left(\frac{\pi}{2}t\right) - 12$$

For each of the following parts, your final answer should **not** include the letters f or g.

a. [2 points] Where is the boat located after 10 seconds?

Solution: At $t = 10$, we have $x = f(10) = 4$ and $y = g(10) = 12\cos(5\pi) - 12 = 12(-1) - 12 = -24$.

Answer: $x =$ 4 and $y =$ -24

b. [3 points] Are there any times during these 10 seconds at which the boat comes to a complete stop? If so, list all such times. If not, write NONE.

Solution:
To find when the boat comes to a complete stop, we look for times when both $f'(t) = 0$ and $g'(t) = 0$.
Since $f(t)$ has a local maximum at $t = 4$, it follows that $f'(4) = 0$, and we observe that this is the only value of t for which $f'(t) = 0$. Furthermore, since

$$g'(t) = -6\pi\sin\left(\frac{\pi}{2}t\right),$$

we have $g'(4) = -6\pi\sin(2\pi) = 0$.
Therefore, the boat comes to a complete stop only at $t = 4$.

Answer: $t =$ 4

c. [4 points] Write an expression involving one or more integrals for the total distance traveled by the boat during the **first 3 seconds**. Do not evaluate any integrals in your answer.

Solution: Computing the slope of the line $x = f(t)$ for $0 \le t \le 3$, we note that

$$f'(t) = \frac{2}{3}, \qquad 0 \le t \le 3.$$

Additionally, we have

$$g'(t) = -6\pi \sin\left(\frac{\pi}{2}t\right).$$

Therefore, the total distance traveled by the boat during the first 3 seconds is given by

$$\int_0^3 \sqrt{\left(\frac{2}{3}\right)^2 + \left(-6\pi\sin\left(\frac{\pi}{2}t\right)\right)^2}\, \mathrm{d}t.$$

Answer: $\int_0^3 \sqrt{\frac{4}{9} + 36\pi^2 \sin^2\left(\frac{\pi}{2}t\right)}\, \mathbf{d}t.$

d. [4 points] What is the tangent line to the boat's path at $t = 9$? Give your answer in cartesian form.

Solution: Note that

$$f(9) = 2, \qquad \text{and} \qquad g(9) = 12\cos\left(\frac{9\pi}{2}\right) - 12 = -12.$$

Also,

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=9} = f'(9) = \frac{4-(-2)}{10-7} = \frac{6}{3} = 2, \quad \text{and} \quad \left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{t=9} = g'(9) = -6\pi\sin\left(\frac{9\pi}{2}\right) = -6\pi.$$

Therefore, the equation of the tangent line to the boat's path at $t = 9$ is

$$(y - (-12)) = \frac{-6\pi}{2}(x-2) \implies y = -3\pi(x-2) - 12.$$

Answer: $y =$ $-3\pi(x-2) - 12$

2. [6 points] Compute the **exact** value of each of the following, if possible. Your answers should not involve integration signs, ellipses or sigma notation. For any values which do not exist, write **DNE**. You do not need to show work.

a. [2 points] The value of $G'(2)$ if $G(x) = \displaystyle\int_1^{3-x} e^{t^3}\,dt$.

Solution: Note that

$$G'(x) = e^{(3-x)^3}(-1) = -e^{(3-x)^3}$$

Therefore, $G'(2) = -e^{(3-2)^3} = -e^{1^3} = -e$.

Answer: $-e$

b. [2 points] The infinite sum $-1 + \frac{5^2}{2!} - \frac{5^4}{4!} + \frac{5^6}{6!} - \cdots + \frac{(-1)^{n+1}5^{2n}}{(2n)!} + \cdots$.

Solution: Using the Taylor series expansion for $\cos(x)$, we obtain

$$\cos(5) = 1 - \frac{5^2}{2!} + \frac{5^4}{4!} - \frac{5^6}{6!} + \cdots + \frac{(-1)^n 5^{2n}}{(2n)!} + \cdots$$

Hence, the value of the infinite sum above is $-\cos(5)$.

Answer: $-\cos(5)$

c. [2 points] The infinite sum $\displaystyle\sum_{n=0}^{\infty} 3(4^n)$.

Solution: This sum is an infinite geometric series with a common ratio 4. Therefore, the series diverges.

Answer: **DNE**

3. [8 points] The two parts of this problem ask about **the same** series. No justification is required for your answers.

a. [4 points] Which of the following series converge? Circle **all** options that apply.

i. $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}}$ iii. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^{1/2}}$ v. $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}}$ vii. NONE OF THESE

ii. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+3}$ iv. $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{(-4)^n}{5^n}}$ vi. $\boxed{\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n\ln(n)}}$

b. [4 points] Which of the following series converge **conditionally**? Circle **all** options that apply.

i. $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}}$ iii. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^{1/2}}$ v. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ vii. NONE OF THESE

ii. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+3}$ iv. $\displaystyle\sum_{n=1}^{\infty} \frac{(-4)^n}{5^n}$ vi. $\boxed{\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n\ln(n)}}$

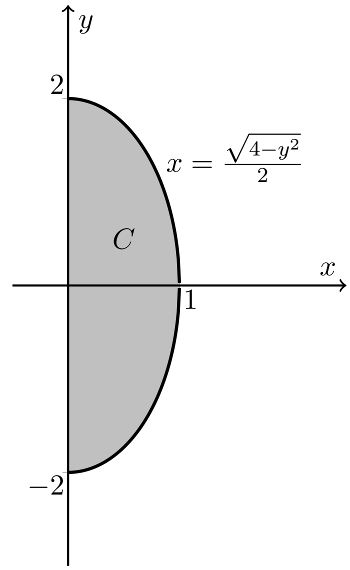
4. [6 points]

Mike owns Mike's Sweet Haven, a bakery that specializes in elegant, custom-made baked goods. With summer approaching, he decides to try something new.

He plans to create a new type of chocolate using the shaded region C, which is bounded by the curves

$$x = \frac{\sqrt{4-y^2}}{2}, \text{ and } x = 0.$$

as illustrated to the right.



a. [4 points] Write an expression involving one or more integrals for the volume of the chocolate obtained by revolving the region C about the y-axis. **Do not** evaluate any integrals in your expression.

Solution: We use horizontal slices to solve this problem. Consider a thin horizontal slice of region C, located y units above the x-axis, with a small thickness Δy. When this slice is rotated about the y-axis, it forms a disk. The inner radius of the disk, denoted by r, is given by

$$r = \frac{\sqrt{4-y^2}}{2}.$$

The approximate volume of this disk is

$$\pi r^2 \Delta y = \frac{\pi(4-y^2)\Delta y}{4}.$$

By integrating from $y = -2$ to $y = 2$, we obtain the total volume of the chocolate:

$$V = \int_{-2}^{2} \frac{\pi(4-y^2)}{4}\, \mathrm{d}y.$$

b. [2 points] If Mike were to take a TRAP(4) estimate of the integral you obtained in part **a.**, would he get an underestimate or an overestimate of the volume of the chocolate? No justification is required.

Circle one: $\boxed{\textbf{Underestimate}}$ **Overestimate**

5. [12 points] The Taylor series centered at $x = -1$ for a function $f(x)$ is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{9^n (n!)^2}{(2n+1)!}(x+1)^{2n+1}$$

a. [7 points] Determine the **radius** of convergence of the Taylor series above. Show all of your work. You do **not** need to find the interval of convergence.

Solution: We use the ratio test to find the radius of convergence. First, we form

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \frac{9^{n+1}((n+1)!)^2|(x+1)^{2(n+1)+1}|}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{9^n(n!)^2|(x+1)^{2n+1}|} \\ &= \frac{9^{n+1}}{9^n} \cdot \frac{((n+1)!)^2}{(n!)^2} \cdot \frac{(2n+1)!}{(2n+3)!} \cdot \frac{|x+1|^{2n+3}}{|x+1|^{2n+1}} \\ &= 9 \cdot (n+1)^2 \cdot \frac{1}{(2n+3)(2n+2)} \cdot |x+1|^2 \end{aligned}$$

Now, we evaluate the limit:

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{9(n+1)^2}{(2n+3)(2n+2)}|x+1|^2 \\ &= \lim_{n\to\infty} \frac{9n^2}{4n^2}|x+1|^2 \\ &= \frac{9}{4}|x+1|^2 \end{aligned}$$

The ratio test tells us that the Taylor series converges when this value is less than 1, i.e., $\frac{9}{4}|x+1|^2 < 1$. Rearranging the inequality, we find that $|x+1|^2 < \frac{4}{9}$, which implies that the radius of convergence is $\frac{2}{3}$.

Answer: $\frac{2}{3}$

b. [5 points] Find $f^{(2025)}(-1)$ and $f^{(2026)}(-1)$. You do not need to simplify your answers.

Solution: All even powers of $(x+1)$ have zero coefficient. Hence $f^{(2026)}(-1) = 0$.
On the other hand, $(x+1)^{2025}$ appears when $2n+1 = 2025$, i.e., when $n = 1012$. Thus

$$\frac{f^{(2025)}(-1)}{2025!} = \frac{9^{1012}(1012!)^2}{2025!}.$$

Rearranging and simplifying, we get

$$f^{(2025)}(-1) = 9^{1012}(1012!)^2.$$

Answer: $f^{(2025)}(-1) =$ $9^{1012}(1012!)^2$ and $f^{(2026)}(-1) =$ 0

6. [10 points] A power series centered at $x = 3$ is given by

$$\sum_{n=1}^{\infty} \frac{2n+1}{5^n(n^2+1)}(x-3)^n.$$

The radius of convergence of this power series is 5 (do **not** show this). Find the **interval** of convergence of this power series. Show all your work, including full justification for series behavior.

Solution: Since we know the radius of convergence, we just need to test the behavior at the endpoints, which are $3 - 5 = -2$, and $3 + 5 = 8$.

At $x = -2$, the series is

$$\sum_{n=1}^{\infty} \frac{2n+1}{5^n \cdot (n^2+1)}(-5)^n = \sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{n^2+1}.$$

To determine the behavior of this, we use the Alternating Series Test. Set $a_n = \dfrac{2n+1}{n^2+1}$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n+1}{n^2+1} = \lim_{n\to\infty} \frac{2n}{n^2} = \lim_{n\to\infty} \frac{2}{n} = 0.$$

and for all $n \geq 1$,

$$0 < a_{n+1} < a_n,$$

so by the Alternating Series test, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{n^2+1}$ converges.
Therefore $x = -2$ is included in the interval of convergence.

At $x = 8$, the series is

$$\sum_{n=1}^{\infty} \frac{2n+1}{5^n \cdot (n^2+1)}(5)^n = \sum_{n=1}^{\infty} \frac{2n+1}{n^2+1}.$$

We can show that this series diverges using either the Direct Comparison Test or the Limit Comparison Test; here, we'll use the Direct Comparison Test.
For $n \geq 1$, $\dfrac{2n+1}{n^2+1} > \dfrac{2n}{n^2+n^2} = \dfrac{1}{n}$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, by the p-test, with $p = 1$. Therefore, by the (Direct) Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2+1}$ diverges. This tells us that $x = 8$ is not included in the interval of convergence.
Therefore, the interval of convergence is $[-2, 8)$.

Interval of convergence: $[-2, 8)$

7. [10 points] Consider the function

$$g(x) = \frac{1}{3}\cos(x^2) - x\sin(x).$$

a. [5 points] Give the first three non-zero terms of the Taylor series of $g(x)$ centered about $x = 0$. Show all your work.

Solution: Using the Taylor series expansion of $\cos(y)$ with $y = x^2$ about $x = 0$, we obtain

$$\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \cdots$$

Therefore, the Taylor series of $\frac{1}{3}\cos(x^2)$ about $x = 0$ is

$$\frac{1}{3}\cos(x^2) = \frac{1}{3}\left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \cdots\right) = \frac{1}{3} - \frac{x^4}{3\cdot 2!} + \frac{x^8}{3\cdot 4!} - \cdots = \frac{1}{3} - \frac{x^4}{6} + \frac{x^8}{3\cdot 4!} - \cdots$$

The Taylor series for $\sin(x)$ about $x = 0$ is given by

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Hence, the Taylor series of $x\sin(x)$ about $x = 0$ is

$$x\sin(x) = x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = x^2 - \frac{x^4}{6} + \frac{x^6}{5!} - \cdots$$

Therefore, the Taylor series of $g(x)$ centered at $x = 0$ is

$$g(x) = \left(\frac{1}{3} - \cancel{\frac{x^4}{6}} + \frac{x^8}{3\cdot 4!} - \cdots\right) - \left(x^2 - \cancel{\frac{x^4}{6}} + \frac{x^6}{5!} - \cdots\right) = \frac{1}{3} - x^2 - \frac{x^6}{5!} + \cdots$$

Answer: $\frac{1}{3} - x^2 - \frac{x^6}{5!} + \cdots$

b. [5 points] The function $g(x)$ has a continuous antiderivative, $G(x)$, with a Taylor series that converges for all x. Given that $G(0) = 8$, find the first four non-zero terms of the Taylor series for $G(x)$ centered about $x = 0$. Show all your work.

Solution: The Second Fundamental Theorem of Calculus implies

$$G(x) = \int_0^x g(t)\,\mathrm{d}t + 8.$$

Using the Taylor series expansion of $g(t)$ from part **a**, and integrating term by term, we obtain the Taylor series expansion of $G(x)$ about $x = 0$:

$$\begin{aligned} G(x) &= 8 + \int_0^x \frac{1}{3}\,\mathrm{d}t - \int_0^x t^2\,\mathrm{d}t - \int_0^x \frac{t^6}{5!}\,\mathrm{d}t + \cdots \\ &= 8 + \frac{x}{3} - \frac{x^3}{3} - \frac{x^7}{7\cdot 5!} + \cdots \end{aligned}$$

Answer: $8 + \frac{x}{3} - \frac{x^3}{3} - \frac{x^7}{7\cdot 5!} + \cdots$

8. [6 points] A team of miners is working to extract a box of minerals from a deep pit. The box weighs 40 lbs, and the rope used to lift it weighs 3 lbs per foot. Initially, when the box is at the bottom of the pit, the rope is 60 feet long. As the box is lifted, the miners do not need to lift the portion of the rope that has already been "reeled in", that is, the part that has reached the top of the pit.

a. [3 points] At a certain moment, the box has already been lifted h feet above the ground. Find an expression for the total weight, in pounds, of the box together with the attached rope that has not yet been reeled in.

Solution: We denote by $F(h)$ the total weight, in pounds, of the box together with the portion of rope that has not yet been reeled in at the instant when the box is h feet above the ground. Because the rope was originally 60 feet long, the length still hanging is $60 - h$ feet. Multiplying this length by the rope's weight density and then adding the weight of the box gives

$$F(h) = 3(60 - h) + 40.$$

Answer: $3(60 - h) + 40$

b. [3 points] Using your expression from part (a), find an expression involving one or more integrals that represents the total work done on the box and attached rope, in foot-pounds, to lift the box from the base of the pit to a point 35 feet above its original position. Do not evaluate any integrals that appear in your answer.

Solution: Suppose the box has already been lifted to a height h feet above the ground. The work required to lift it an additional Δh feet is given by:

$$F(h)\,\Delta h = (3(60 - h) + 40)\,\Delta h.$$

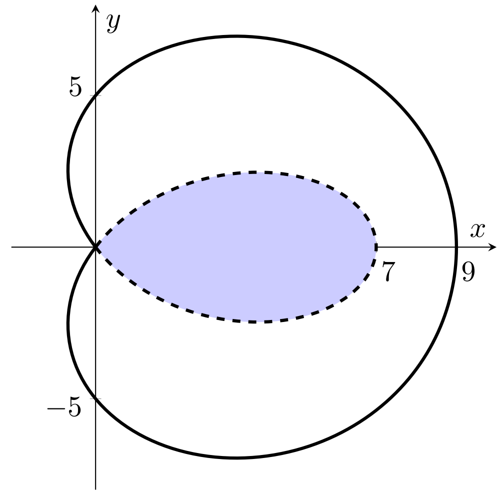
Therefore, the total work required to lift the box from the bottom of the pit to a point 35 feet above its initial position is given by

$$W = \int_0^{35} (3(60 - h) + 40)\ \mathrm{d}h.$$

Answer: $\int_0^{35} (3(60 - h) + 40)\ \mathbf{d}h$

9. [12 points]

Elena, a talented landscape architect, envisions a park whose shape is defined by the polar curve $r = 5 + 8\cos(\theta) - 4\cos^2(\theta)$, as illustrated to the right. In her design, the inner loop of the curve serves as an ideal location for a lake, represented by the shaded region. The solid outer curve in the diagram represents the walking trail that winds around the park.



a. [4 points] Using the factorization $5 + 8\cos(\theta) - 4\cos^2(\theta) = (1 + 2\cos(\theta))(5 - 2\cos(\theta))$, find the values of θ in the interval $[0, 2\pi)$ for which the curve passes through the origin.

Solution: To find the values of θ in the interval $[0, 2\pi)$ where the curve passes through the origin, we set $r = 0$. Using the factorization $5+8\cos(\theta)-4\cos^2(\theta) = (1+2\cos(\theta))(5-2\cos(\theta))$, we find that $r = 0$ when

$$\begin{aligned} 1 + 2\cos(\theta) = 0 \quad &\text{or} \quad 5 - 2\cos(\theta) = 0 \\ \cos(\theta) = -\frac{1}{2} \quad &\text{or} \quad \cos(\theta) = \frac{5}{2}. \end{aligned}$$

The equation $\cos(\theta) = -\frac{1}{2}$ has two solutions in the interval $[0, 2\pi)$: $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$. The equation $\cos(\theta) = \frac{5}{2}$ has no solutions.

Answer: $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

b. [4 points] To determine the amount of water required to fill the lake, Elena wants to calculate the area of the surface of the lake. Write an expression involving one or more integrals that represents the area of the shaded region. Do not evaluate the integral(s).

Solution: From part **a**, we determine that the inner loop is traced for θ in the interval $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$. Using the formula for the area, the total area of the shaded region is given by

$$\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left(5 + 8\cos(\theta) - 4\cos^2(\theta)\right)^2 \, d\theta.$$

Answer: $\displaystyle\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{(5 + 8\cos(\theta) - 4\cos^2(\theta))^2}{2} \, \mathbf{d\theta}$

c. [4 points] Recall that the solid outer curve in the diagram represents the walking trail. Write an expression involving one or more integrals that represents the total length of the walking trail. Do not evaluate the integral(s).

Solution: The solid outer curve is traced for θ in the intervals $\left(0, \frac{2\pi}{3}\right)$ and $\left(\frac{4\pi}{3}, 2\pi\right)$. By symmetry, both segments have equal lengths. Note that

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -8\sin(\theta) + 8\cos(\theta)\sin(\theta)$$

Using the formula for arc length, the total length of the walking trail is given by

$$2\int_0^{\frac{2\pi}{3}} \sqrt{\left(-8\sin(\theta) + 8\cos(\theta)\sin(\theta)\right)^2 + \left(5 + 8\cos(\theta) - 4\cos^2(\theta)\right)^2}\,\mathrm{d}\theta$$

Answer: $2\int_0^{\frac{2\pi}{3}} \sqrt{(-8\sin(\theta) + 8\cos(\theta)\sin(\theta))^2 + \left(5 + 8\cos(\theta) - 4\cos^2(\theta)\right)^2}\,\mathbf{d}\theta$

10. [9 points] Let a be a real number. Consider the following integral

$$\int_0^1 ax\ln(x)\,\mathrm{d}x$$

a. [8 points] Show that the above integral converges by using a **direct computation** and find its value in terms of a. Be sure to show your full computation, and be sure to use **proper notation**.

Solution: By definition,

$$\int_0^1 x\ln(x)\,\mathrm{d}x = \lim_{b\to 0^+}\int_b^1 x\ln(x)\,\mathrm{d}x$$

We set $u = \ln(x)$, $\mathrm{d}v = x$. Then, $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$, $v = \frac{x^2}{2}$. Now, integrating by parts,

$$\begin{aligned}
\lim_{b\to 0^+}\int_b^1 x\ln(x)\,\mathrm{d}x &= \lim_{b\to 0^+}\left(\frac{x^2\ln(x)}{2}\bigg|_b^1 - \int_b^1 \frac{x}{2}\,\mathrm{d}x\right)\\
&= \lim_{b\to 0^+}\left(\left(\frac{1^2\ln(1)}{2} - \frac{b^2\ln(b)}{2}\right) - \frac{x^2}{4}\bigg|_b^1\right)\\
&= \lim_{b\to 0^+}\left(-\frac{b^2\ln(b)}{2} - \left(\frac{1}{4} - \frac{b^2}{4}\right)\right)\\
&= \lim_{b\to 0^+}\frac{-\ln(b)}{\frac{2}{b^2}} - \frac{1}{4}\\
&\overset{\text{L'H}}{=} \lim_{b\to 0^+}\frac{-\frac{1}{b}}{-\frac{4}{b^3}} - \frac{1}{4}\\
&= \lim_{b\to 0^+}\frac{b^2}{4} - \frac{1}{4}\\
&= -\frac{1}{4}.
\end{aligned}$$

Therefore,

$$\int_0^1 ax\ln(x)\,\mathrm{d}x = a\int_0^1 x\ln(x)\,\mathrm{d}x = -\frac{a}{4}.$$

Answer: $\int_0^1 ax\ln(x)\,\mathrm{d}x =$ $-\frac{a}{4}$

b. [1 point] Find the value of a so that the function

$$p(x) = \begin{cases} ax\ln(x) & 0 < x < 1\\ 0 & \text{otherwise}\end{cases}$$

is a probability density function (pdf).

Solution: For the function $p(x)$ to be a probability density function, it must satisfy $p(x) \geq 0$ and $\int_0^1 ax\ln(x)\,\mathrm{d}x = 1$. From part **a.**, we find that a must satisfy $-\frac{a}{4} = 1$, which implies that $a = -4$.

Answer: $a =$ -4

11. [8 points] The parts of this problem are unrelated. No justification is required for your answers.

a. [2 points] Let $p(x)$ be a probability density function (pdf). Then, which of the following functions must also be probability density function? Circle **all** options which apply.

i. $2p(x)$

ii. $p(2x)$

iii. $\boxed{2\,p(2x)}$

iv. $2x\,p(x^2)$

v. $\boxed{3x^2\,p(x^3)}$

vi. NONE OF THESE

b. [2 points] The cartesian coordinates of a point A are $(x, y) = \left(\sqrt{2}, -\sqrt{2}\right)$. Which of the following represent the polar coordinates of point A? Circle **all** options which apply.

i. $(r, \theta) = \left(\sqrt{2}, -\frac{\pi}{4}\right)$

ii. $\boxed{(r, \theta) = \left(2, -\frac{\pi}{4}\right)}$

iii. $\boxed{(r, \theta) = \left(2, \frac{7\pi}{4}\right)}$

iv. $(r, \theta) = \left(-2, \frac{\pi}{4}\right)$

v. $\boxed{(r, \theta) = \left(-2, \frac{3\pi}{4}\right)}$

vi. NONE OF THESE

c. [2 points] A power series $\displaystyle\sum_{n=0}^{\infty} C_n(x-a)^n$ converges at $x = -4$ and diverges at $x = 2$. Which of the following values could be the center, a, of the power series? Circle **all** options which apply.

i. $\boxed{a = -2}$

ii. $\boxed{a = -1}$

iii. $a = 0$

iv. $a = 1$

v. $a = 2$

vi. NONE OF THESE

d. [2 points] At which of the following values of θ in $[0, \pi)$ does the curve $r = \cos(\theta)$ have a horizontal tangent line? Circle **all** options which apply.

i. $\theta = 0$

ii. $\boxed{\theta = \frac{\pi}{4}}$

iii. $\theta = \frac{\pi}{3}$

iv. $\theta = \frac{\pi}{2}$

v. $\boxed{\theta = \frac{3\pi}{4}}$

vi. NONE OF THESE

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \le 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$

Select Values of Trigonometric Functions:

θ	$\sin\theta$	$\cos\theta$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$