

Math 116 — First Midterm — February 9, 2026

EXAM SOLUTIONS

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1. Please neatly write your 8-digit UMID number, your initials, your instructor's first and/or last name, and your section number in the spaces provided.
2. This exam has 15 pages including this cover.
3. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	15	
2	15	
3	13	
4	7	
5	9	

Problem	Points	Score
6	10	
7	10	
8	11	
9	10	
Total	100	

1. [15 points] Consider the differentiable function $f(x)$ defined for $x \geq 0$ by

$$f(x) = 2 \ln(4x + e).$$

- a. [3 points] Which of the following functions are antiderivatives of $f(x)$? Circle **all** that apply.

- i. $\frac{1}{2}(\ln(4x + e))^2$ iii. $\int_0^1 2 \ln(4x + e) dx$ v. $\frac{1}{2} \int_1^{4x+e} \ln(s) ds$
 ii. $\frac{1}{2}(4x + e) \ln(4x + e) - 2x$ iv. $1 + \int_0^x 2 \ln(4t + e) dt$ vi. None of these

- b. [2 points] Suppose that $F(x)$ is an antiderivative of $f(x)$. Which of the following quantities **must** be equal to the average value of the function $f(x)$ on the interval $[2, 10]$? Circle **all** that apply.

- i. $\frac{F(10) - F(2)}{8}$ ii. $\frac{1}{8} \int_2^{10} f(x) dx$ iii. $\frac{1}{8} \int_2^{10} f'(x) dx$ iv. $\frac{f(10) - f(2)}{8}$

- c. [4 points] Set up, but do not evaluate, a definite integral that represents the arclength of the curve $f(x)$ between $x = 0$ and $x = 8$. Your answer should **not** involve the letter f .

Solution: First, we take the derivative of $f(x)$ for $x \geq 0$:

$$f'(x) = \frac{8}{4x + e}$$

Using the arc length formula, we get

$$\text{Arc length} = \int_0^8 \sqrt{1 + (f'(x))^2} dx = \int_0^8 \sqrt{1 + \left(\frac{8}{4x + e}\right)^2} dx = \int_0^8 \sqrt{1 + \frac{64}{(4x + e)^2}} dx$$

Answer: $\int_0^8 \sqrt{1 + \frac{64}{(4x + e)^2}} dx$

- d. [3 points] Approximate the integral $\int_0^8 f(x) \, dx$ using MID(4). Write out all the terms in your sum. You do not need to simplify, but your final answer should **not** involve the letter f .

Solution: The interval of integration is given by $[0, 8]$, therefore, $\Delta x = \frac{8-0}{4} = 2$. The subintervals are given by $[0, 2]$, $[2, 4]$, $[4, 6]$, $[6, 8]$, hence,

$$\begin{aligned} \text{MID}(4) &= \Delta x \cdot (f(1) + f(3) + f(5) + f(7)) \\ &= 2(2 \ln(4 + e) + 2 \ln(12 + e) + 2 \ln(20 + e) + 2 \ln(28 + e)) \\ &= 4(\ln(4 + e) + \ln(12 + e) + \ln(20 + e) + \ln(28 + e)) \end{aligned}$$

Answer: $4(\ln(4 + e) + \ln(12 + e) + \ln(20 + e) + \ln(28 + e))$

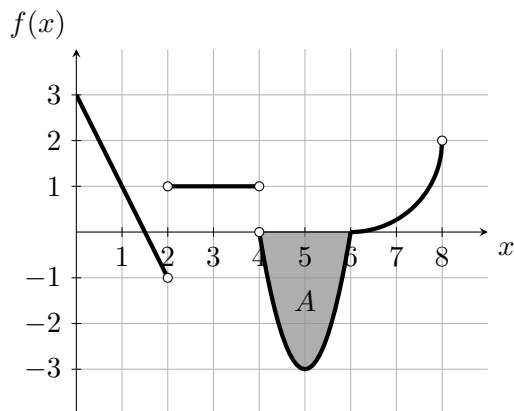
- e. [3 points] Is your answer to part d. an overestimate or underestimate to $\int_0^8 f(x) \, dx$? Briefly explain your reasoning.

Solution: We check the concavity of the function $f(x)$:

$$f''(x) = \frac{d}{dx} \left(\frac{8}{4x + e} \right) = -8 \frac{4}{(4x + e)^2} < 0 \text{ for all } x \geq 0$$

Therefore, $f(x)$ is concave down, and MID(4) is an overestimate of the integral.

2. [15 points] An **even** function $f(x)$ is graphed below. The area of the shaded region A is 4. On $[6, 8]$, the graph of $f(x)$ is a quarter circle with radius 2 centered at $(6, 2)$.



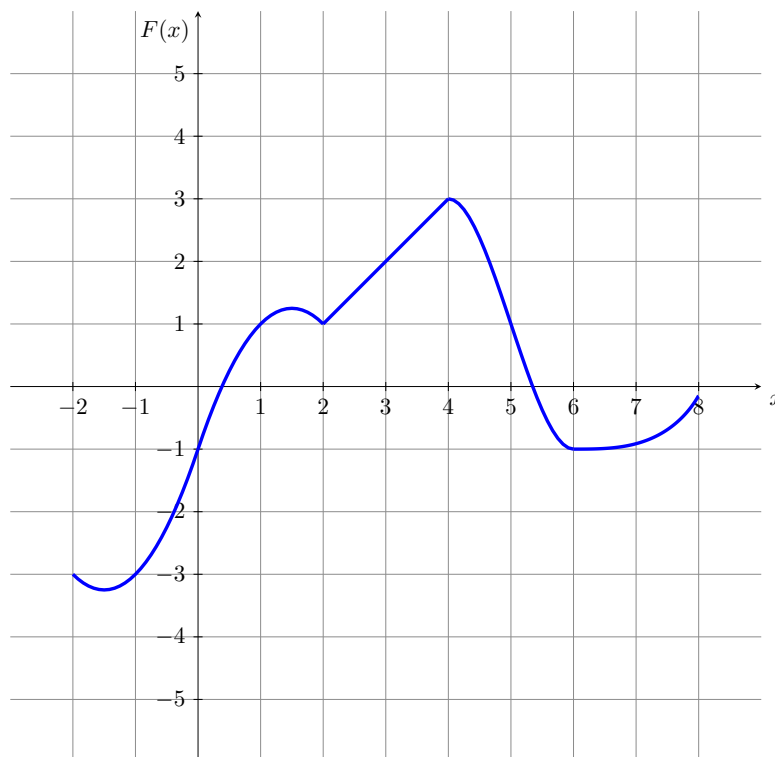
Let $F(x)$ be a continuous antiderivative of $f(x)$ satisfying $F(2) = 1$.

- a. [5 points] Use the graph of $f(x)$ to complete the table below with the exact values of $F(x)$.

x	-2	0	2	4	6	8
$F(x)$	-3	-1	1	3	-1	$3 - \pi$

- b. [10 points] On the axis below, sketch a graph of $F(x)$ on the interval $[-2, 8]$. Be sure to pay attention to:

- the values of $F(x)$ you found in part (a).
- where $F(x)$ is increasing, decreasing, or constant.
- where $F(x)$ is not differentiable.
- where $F(x)$ is concave up, concave down, or linear.



c. [5 points] $\int_4^9 e^{\sqrt{x}} dx$

Solution: First, use the substitution $u(x) = \sqrt{x}$. Note $du = \frac{1}{2\sqrt{x}} dx$. Hence, we have

$$dx = 2\sqrt{x} du = 2u du$$

Then apply integration by parts to the integral in the u -domain:

$$\begin{aligned} & \int_4^9 e^{\sqrt{x}} dx \\ &= \int_2^3 2ue^u du \\ &= 2e^u u \Big|_2^3 - \int_2^3 2e^u du \\ &= 2(3e^3 - 2e^2 - e^3 + e^2) \\ &= 4e^3 - 2e^2 \end{aligned}$$

Answer: $4e^3 - 2e^2$

4. [7 points] Use the partial fraction decomposition

$$\frac{4x^2 + 11x - 9}{(5 - 2x)(x^2 + 1)} = \frac{6}{5 - 2x} + \frac{x - 3}{x^2 + 1}$$

to evaluate the following indefinite integral, showing all of your work.

$$\int \frac{4x^2 + 11x - 9}{(5 - 2x)(x^2 + 1)} dx$$

Solution:

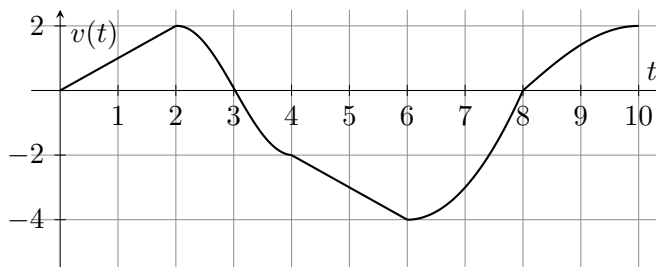
$$\begin{aligned} \int \frac{4x^2 + 11x - 9}{(5 - 2x)(x^2 + 1)} dx &= \int \frac{6}{5 - 2x} + \frac{x - 3}{x^2 + 1} dx \\ &= 6 \int \frac{1}{5 - 2x} dx + \int \frac{x}{x^2 + 1} dx - 3 \int \frac{1}{x^2 + 1} dx \\ &= -3 \ln |5 - 2x| + \frac{1}{2} \ln |x^2 + 1| - 3 \arctan x + C \end{aligned}$$

Here, we have applied the method of substitution to evaluate the first two integrals in the second step.

Answer: $\underline{-3 \ln |5 - 2x| + \frac{1}{2} \ln |x^2 + 1| - 3 \arctan x + C}$

5. [9 points]

Carmen leads a team who operate a new Mars rover, named “*Determination*”. At $t = 0$, the rover starts moving, and the function $v(t)$ gives the velocity of the rover t seconds later. Assume that the rover travels on flat ground, in a straight line, and that $v(t)$ is measured in feet per second. A graph of $v(t)$, for $0 \leq t \leq 10$, is shown to the right.



- a. [2 points] Estimate the time t with $0 \leq t \leq 10$ at which the rover is furthest from its initial position.

Answer: $t =$ 8

- b. [2 points] Find an expression involving one or more integrals which gives the distance, in feet, between the rover's initial position, and its position 10 seconds later. Your answer **may** involve the letters v and t .

Answer: $-\int_0^{10} v(t) dt$

- c. [2 points] Find an expression involving one or more integrals which gives the total distance, in feet, traveled by the rover in these 10 seconds. Your answer **may** involve the letters v and t .

Answer: $\int_0^{10} |v(t)| dt$

- d. [3 points] The rate of change of the temperature of the rover m minutes after 12pm noon (Carmen's time) is given by the function $r(m)$, where $r(m)$ is measured in degrees Celsius per minute ($^{\circ}\text{C}/\text{min}$). Carmen finds left and right estimates to $\int_0^{120} r(m) dm$, and finds that $\text{LEFT}(6) = 6$ and $\text{RIGHT}(6) = 8$. Which of the following **must** be true? Circle **all** correct answers.

i. The value $r(120)$ is larger than the value $r(0)$.

ii. The function $r(m)$ is increasing on the entire interval $[0, 120]$.

iii. The temperature of the rover at 2pm is at least 6°C higher than at 12pm noon.

iv. The temperature of the rover at 2pm is no more than 8°C higher than at 12pm noon.

v. The rate of change of the temperature of the rover at 2pm is $2^{\circ}\text{C}/\text{min}$ higher than it was at noon.

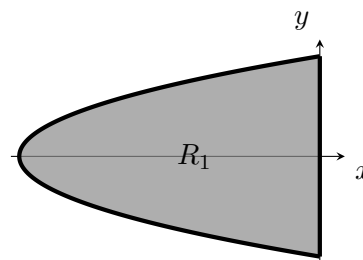
vi. The rate of change of the temperature of the rover at 2pm is $0.1^{\circ}\text{C}/\text{min}$ higher than it was at noon.

vii. None of the above.

6. [10 points] Justin, a member of the 2023 undefeated Michigan football team, is graduating this semester. He wants to celebrate by building a replica of the Rose Bowl trophy that is twice the size of the original. The trophy consists of two components: a silver football and a plinth.

a. [5 points]

Let R_1 be the region bounded by the y -axis and the curve $x = \left(\frac{y}{2}\right)^2 - 36$, as shown to the right. To form the silver football, Justin rotates this region around the y -axis.



Find an expression involving one or more integrals for the volume of the football using **horizontal** slices of the region R_1 . **Do not** evaluate any integrals in your expression.

Solution: We use horizontal slices to solve this problem. Consider a thin horizontal slice of region R_1 , located at y , with a small thickness Δy . When this slice is rotated about the y -axis, it forms a disk. The radius of the disk, denoted by R , is given by

$$R = 36 - \left(\frac{y}{2}\right)^2$$

The approximate volume of this disk is

$$\pi R^2 \Delta y = \pi \left(36 - \left(\frac{y}{2}\right)^2\right)^2 \Delta y.$$

Note that the domain of the slices are given by the y -intercepts of the curve $\left(\frac{y}{2}\right)^2 - 36$, which yields $y = -12$ and $y = 12$. By integrating from $y = -12$ to $y = 12$, we obtain the total volume of the football:

$$V = \int_{-12}^{12} \pi \left(36 - \left(\frac{y}{2}\right)^2\right)^2 dy.$$

$$\int_{-12}^{12} \pi \left(36 - \left(\frac{y}{2}\right)^2\right)^2 dy$$

Answer: _____

b. [5 points]

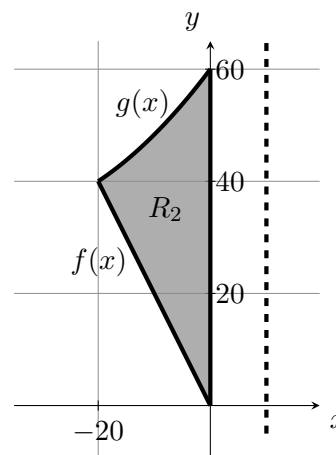
Consider the shaded region R_2 depicted to the right. The region is bounded by $y = f(x)$, $y = g(x)$, and the y -axis, where

$$g(x) = \frac{1}{60} ((x + 40)^2 + 2000),$$

and

$$f(x) = -2x.$$

The plinth for the trophy is formed by rotating the region R_2 about the vertical line $x = 10$.



Find an expression involving one or more integrals for the volume of the plinth of the trophy using **vertical** slices of the region R_2 . Your answer should not involve the letters f or g . **Do not** evaluate any integrals in your expression.

Solution: We use vertical slices to solve this problem. Consider a thin vertical slice of region R_2 , located x units to the left of the line $x = 10$, with a small thickness Δx . When this slice is rotated about the $x = 10$, it forms a shell. The height and radius of the shell, denoted by h and r , are given by

$$h = g(x) - f(x) = \frac{1}{60} ((x + 40)^2 + 2000) + 2x \quad \text{and} \quad r = 10 - x$$

Hence, the approximate volume of this shell is

$$2\pi r h \Delta x = 2\pi \left(\frac{1}{60} ((x + 40)^2 + 2000) + 2x \right) (10 - x) \Delta x$$

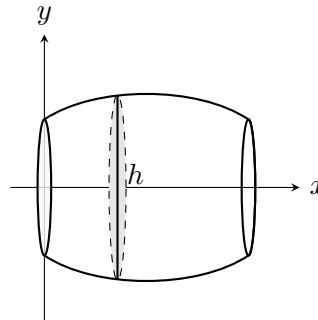
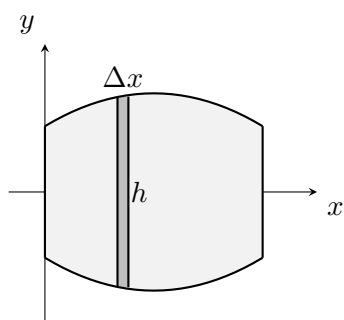
To determine the domain of x , we solve the equation $f(x) = g(x)$, which gives two answers $x = -20$ or $x = 0$. Hence, we obtain the total volume of the plinth:

$$V = \int_{-20}^0 2\pi \left(\frac{1}{60} ((x + 40)^2 + 2000) + 2x \right) (10 - x) dx$$

$$\text{Answer: } \int_{-20}^0 2\pi \left(\frac{1}{60} ((x + 40)^2 + 2000) + 2x \right) (10 - x) dx$$

7. [10 points] Gragas, a brewmaster, gifts his friend Braum a barrel **full** of whiskey. Braum wants to determine the total mass of the alcohol stored inside.

The side view of the barrel, shown below and to the left, is bounded by the vertical lines $x = 0$ and $x = 60$, and the curves $y = 30 - \frac{1}{90}(x - 30)^2$ and $y = -30 + \frac{1}{90}(x - 30)^2$. All lengths are measured in centimeters.



- a. [2 points] Consider a thin vertical strip of the barrel located x cm from the y -axis, as shown in the diagram above and to the left. This strip has a vertical length h . Find a formula for h in terms of x .

Solution: The length h is given by the difference of the top and bottom curves:

$$h = 30 - \frac{1}{90}(x - 30)^2 - \left(-30 + \frac{1}{90}(x - 30)^2\right) = 60 - \frac{2}{90}(x - 30)^2$$

$$60 - \frac{2}{90}(x - 30)^2$$

Answer: $h =$ _____

- b. [4 points] Cross-sections of the barrel perpendicular to the x -axis are circles. Consider a thin cylindrical slice located x cm from the y -axis, with thickness Δx . Find an expression for the volume of that slice. Your answer should **not** involve any integrals, and should **not** include the letter h . Include units.

Solution: Note that the radius of the cylindrical slice, denoted by r , is given by

$$r = \frac{h}{2} = 30 - \frac{1}{90}(x - 30)^2$$

Therefore, the approximate volume of the slice is given by

$$\pi r^2 \Delta x = \pi \left(30 - \frac{1}{90}(x - 30)^2\right)^2 \Delta x$$

Answer: _____ $\pi \left(60 - \frac{2}{90}(x - 30)^2\right)^2 \Delta x$ **Units:** _____ cm^3

- c. [4 points] Braum determines that the density of the liquid varies along the length of the barrel according to the function $\rho(x) = 0.01x + 1$, measured in g/cm^3 .

Write an expression involving one or more integrals that represents the total mass of the whiskey inside the barrel. **Do not** evaluate any integrals in your expression. Your answer should **not** involve the letters h or ρ . Include units.

Solution: To find the mass of the cylindrical slice from part b. we multiply its volume by the density function. We see its mass is

$$(0.01x + 1)\pi \left(60 - \frac{2}{90}(x - 30)^2\right)^2 \Delta x$$

Integrating from $x = 0$ to $x = 60$, we get the total mass

$$m = \int_0^{60} (0.01x + 1)\pi \left(30 - \frac{1}{90}(x - 30)^2\right)^2 dx$$

Answer: $\int_0^{60} (0.01x + 1)\pi \left(30 - \frac{1}{90}(x - 30)^2\right)^2 dx$

Units: g

8. [11 points] Consider the following function $y(t)$ defined for $t \geq 0$ by:

$$y(t) = 3 + e^{-t} \int_0^{\sin(t)} \frac{1}{1+x^4} dx$$

- a. [2 points] Let $y_0 = y(\pi)$. Determine the value of y_0 .

Solution: Evaluate $y(t)$ at $t = \pi$, we get:

$$y(\pi) = 3 + e^{-\pi} \int_0^{\sin(\pi)} \frac{1}{1+x^4} dx = 3 + e^{-\pi} \int_0^0 \frac{1}{1+x^4} dx = 3 + 0.$$

Answer: $y_0 =$ 3

- b. [4 points] Compute the derivative $y'(t)$. Your answer may involve an integral.

Solution: Apply chain rule and product rule:

$$\begin{aligned} y'(t) &= \frac{d}{dt}(e^{-t}) \int_0^{\sin(t)} \frac{1}{1+x^4} dx + e^{-t} \frac{d}{dt} \left(\int_0^{\sin(t)} \frac{1}{1+x^4} dx \right) \\ &= -e^{-t} \int_0^{\sin(t)} \frac{1}{1+x^4} dx + e^{-t} \frac{1}{1+\sin(t)^4} \cos(t) \end{aligned}$$

Answer: $y'(t) =$ $-e^{-t} \int_0^{\sin(t)} \frac{1}{1+x^4} dx + e^{-t} \frac{\cos(t)}{1+\sin(t)^4}$

- c. [2 points] The function $y(t)$ is a *solution* to a first-order differential equation of the form:

$$e^t y'(t) + e^t y(t) = f(t) \quad \text{for } t \geq 0 \quad (\star)$$

Substitute your expressions for $y(t)$ and $y'(t)$ into the left-hand side of $(\star)$. Simplify the result to find an explicit formula for $f(t)$. Your final expression for $f(t)$ should **not** involve any integrals.

Solution: Use the expression for $y(t)$ and $y'(t)$, we get

$$\begin{aligned} e^t y'(t) + e^t y(t) &= e^t \left(3 + e^{-t} \int_0^{\sin(t)} \frac{1}{1+x^4} dx \right) \\ &\quad + e^t \left(-e^{-t} \int_0^{\sin(t)} \frac{1}{1+x^4} dx + e^{-t} \frac{\cos(t)}{1+\sin(t)^4} \right) \\ &= 3e^t + \frac{\cos(t)}{1+\sin(t)^4} \end{aligned}$$

Answer: $f(t) =$ $3e^t + \frac{\cos(t)}{1+\sin(t)^4}$

d. [3 points] It can be shown that we may rewrite the solution $y(t)$ in the following form:

$$y(t) = e^{-t} \left(\int_a^t f(x) dx + C \right)$$

where a and C are constants, and $f(x)$ is the expression you found in part c. Find values of the constants a and C so that the solution satisfies $y(\pi) = y_0$, where y_0 is the value you found in part a.

Solution: Using the fact $y(\pi) = 3$, we see

$$e^{-\pi} \left(\int_a^{\pi} f(x) dx + C \right) = 3$$

Pick $a = \pi$ to eliminate the integral, we get $e^{-\pi}C = 3$, hence $C = 3e^{-\pi}$.

Answer: $a = \underline{\hspace{1.5cm}\pi\hspace{1.5cm}}$ and $C = \underline{\hspace{1.5cm}3e^{-\pi}\hspace{1.5cm}}$

9. [10 points] For the following questions, determine if the statement is ALWAYS true, SOMETIMES true, or NEVER true, and circle the corresponding answer. Justification is not required.

- a. [2 points] If $f(x)$ is an increasing, differentiable function, and RIGHT(4) is used to estimate $\int_{-2}^2 f(x) dx$, then

$$\int_{-2}^2 f(x) dx \leq \text{RIGHT}(4).$$

Circle one:

ALWAYS

SOMETIMES

NEVER

- b. [2 points] Suppose that $g(x)$ is a concave down, differentiable function, and TRAP(4) is used to estimate $\int_0^4 g(x) dx$. Then

$$\text{TRAP}(4) \geq \int_0^4 g(x) dx.$$

Circle one:

ALWAYS

SOMETIMES

NEVER

- c. [2 points] Suppose $h(x)$ is a concave down, even differentiable function and $a < 0 < b$ are constants. If RIGHT(4) and LEFT(4) are used to estimate $\int_a^b h(x) dx$, then

$$\text{RIGHT}(4) = \text{LEFT}(4).$$

Circle one:

ALWAYS

SOMETIMES

NEVER

- d. [2 points] If $D(x)$ is an increasing, concave up differentiable function, and TRAP(8) and RIGHT(4) are used to estimate $\int_{-2}^2 D(x) dx$, then

$$\int_{-2}^2 D(x) dx < \text{TRAP}(8) < \text{RIGHT}(4).$$

Circle one:

ALWAYS

SOMETIMES

NEVER

- e. [2 points] Suppose that $E(x)$ is a positive, increasing differentiable function, and that LEFT(4) and MID(2) are used to estimate $\int_{-2}^2 E(x) dx$. Then

$$\text{LEFT}(4) \leq \text{MID}(2) \leq \int_{-2}^2 E(x) dx.$$

Circle one:

ALWAYS

SOMETIMES

NEVER