Math 116 — Second Midterm — March 23, 2026

EXAM SOLUTIONS

1. Please neatly write your 8-digit UMID number, your initials, your instructor's first and/or last name, and your section number in the spaces provided.
2. This exam has 14 pages including this cover.
3. There are 11 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	11	
2	4	
3	7	
4	7	
5	10	
6	7	

Problem	Points	Score
7	10	
8	11	
9	12	
10	9	
11	12	
Total	100	

1. [11 points] Suppose $p(t)$ is the **probability density function** for a variable t.

$$p(t) = \begin{cases} 0 & t \leq 0 \\ at^2 & 0 < t \leq \frac{1}{2} \\ t & \frac{1}{2} < t \leq 1 \\ 0 & t > 1 \end{cases}$$

a. [3 points] Find the value of a such that $p(t)$ is a probability density function.

Solution: Since $p(t)$ is a pdf, we know that $\int_{-\infty}^{\infty} p(t)dt = 1$:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} p(t)dt \\ &= \int_0^{\frac{1}{2}} at^2\ dt + \int_{\frac{1}{2}}^1 t\ dt \\ &= \frac{at^3}{3}\bigg|_0^{\frac{1}{2}} + \frac{t^2}{2}\bigg|_{\frac{1}{2}}^1 \\ &= \frac{a}{24} + \frac{1}{2} - \frac{1}{8} \end{aligned}$$

Solve for a to get $a = \frac{5}{8}(24) = 15$.

Answer: $a =$ 15

b. [3 points] Write an expression involving one or more integrals for the mean of the distribution. **Do not evaluate any integrals in your answer**. Your final answer should not include the letters p or P, but may include the letter a.

Solution: The mean of the distribution is given by

$$\int_0^{1/2} at^3\ dt + \int_{1/2}^1 t^2\ dt = \int_0^{1/2} 15t^3\ dt + \int_{1/2}^1 t^2\ dt$$

Answer: $\int_0^{1/2} 15t^3\ dt + \int_{1/2}^1 t^2\ dt$

c. [5 points] Write a formula for the corresponding cumulative distribution function, $P(t)$. Your formula should not contain any integral signs, but may include the letter a.

Solution:
The function $P(t)$ must be a continuous antiderivative of $p(t)$ which satisfies $\lim_{t\to-\infty} P(t) = 0$ and $\lim_{t\to\infty} P(t) = 1$. We have:

$$P(t) = \int_{-\infty}^{t} p(x)\ dx = \begin{cases} 0 & t \le 0 \\ \frac{a}{3}t^3 & 0 < t \le \frac{1}{2} \\ \frac{a}{24} + \frac{t^2}{2} - \frac{1}{8} & \frac{1}{2} < t \le 1 \\ 1 & t > 1 \end{cases}$$

Plugging in $a = 15$,

$$P(t) = \int_{-\infty}^{t} p(x)\ dx = \begin{cases} 0 & t \le 0 \\ 5t^3 & 0 < t \le \frac{1}{2} \\ \frac{1}{2} + \frac{t^2}{2} & \frac{1}{2} < t \le 1 \\ 1 & t > 1 \end{cases}$$

2. [4 points] Compute the following limit. Fully justify your answer and use **proper limit notation**.

$$\lim_{x\to 0} \frac{\sin^2(7x)}{\cos(7x)-1}$$

Solution: We use L'Hospital's Rule:

$$\begin{aligned}\lim_{x\to 0} \frac{\sin^2(7x)}{\cos(7x)-1} &\overset{\text{LH}}{=} \lim_{x\to 0} \frac{14\sin(7x)\cos(7x)}{-7\sin(7x)} \\ &= \lim_{x\to 0} -2\cos(7x) \\ &= -2.\end{aligned}$$

Answer: $\lim_{x\to 0} \frac{\sin^2(7x)}{\cos(7x)-1} =$ -2

3. [7 points] **Compute** the value of the following improper integral if it converges. If it does not converge, use a **direct computation** of the integral to show its divergence. Be sure to show your full computation, and be sure to use **proper notation**.

$$\int_2^\infty \frac{e^{-3/x}}{x^2}\,dx$$

Solution: By definition,

$$\int_2^\infty \frac{e^{-3/x}}{x^2}\,dx = \lim_{b\to\infty}\int_2^b \frac{e^{-3/x}}{x^2}\,dx.$$

Set $u = \frac{3}{x}$. Then $du = -\frac{3}{x^2}\,dx$.
Therefore,

$$\begin{aligned}\int_2^\infty \frac{e^{-3/x}}{x^2}\,dx &= \lim_{b\to\infty}\int_2^b \frac{e^{-3/x}}{x^2}\,dx. \\ &= \lim_{b\to\infty}\int_{3/2}^{3/b} -\frac{1}{3}e^{-u}\,du. \\ &= \lim_{b\to\infty} \frac{1}{3}e^{-u}\Big|_{3/2}^{3/b} \\ &= \lim_{b\to\infty} \frac{1}{3}e^{-3/b} - \frac{1}{3}e^{-3/2} \\ &= \frac{1}{3} - \frac{1}{3}e^{-3/2}\end{aligned}$$

Circle one: **Diverges** $\boxed{\textbf{Converges to } \frac{1}{3}\left(1-e^{-3/2}\right)}$

4. [7 points] Determine whether the following improper integral converges or diverges and circle the corresponding word. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests you use. You do not need to calculate the value of the integral if it converges.

$$\int_0^1 \frac{7}{x^{1/3} + x^{4/3}}\, \mathrm{d}x$$

Circle one: $\boxed{\textbf{Converges}}$ **Diverges**

Justification:

Solution: On the interval $0 \le x \le 1$, we have $\dfrac{7}{x^{1/3} + x^{4/3}} \le \dfrac{7}{x^{1/3}}$, and $\displaystyle\int_0^1 \frac{7}{x^{1/3}}\, \mathrm{d}x$ converges by the p-test with $p = \dfrac{1}{3}$. Therefore, by the (Direct) Comparison Test, $\displaystyle\int_0^1 \frac{7}{x^{1/3} + x^{4/3}}\, \mathrm{d}x$ converges.

5. [10 points] For each of the following parts, suppose that F is a nonnegative function defined for all real numbers with the given property or properties. Circle **all** options that apply to F. Note that "pdf" and "cdf" stand for "probability density function" and "cumulative distribution function", respectively. No justification is required.

a. [2 points] $\int_{-\infty}^{\infty} F(x)dx = 1$ and $F(3) = 4$.

i. $\boxed{F \text{ could be a pdf.}}$

ii. F could be a cdf.

iii. F is definitely not a pdf.

iv. $\boxed{F \text{ is definitely not a cdf.}}$

b. [2 points] $\lim_{x\to\infty} F(x) = 1$ and $F(3) < F(2)$.

i. F could be a pdf.

ii. F could be a cdf.

iii. $\boxed{F \text{ is definitely not a pdf.}}$

iv. $\boxed{F \text{ is definitely not a cdf.}}$

c. [2 points] $F'(x) > 0$ for $x \geq 0$.

i. F could be a pdf.

ii. $\boxed{F \text{ could be a cdf.}}$

iii. $\boxed{F \text{ is definitely not a pdf.}}$

iv. F is definitely not a cdf.

d. [2 points] $\sum_{n=0}^{\infty} F(-n) = 1$.

i. $\boxed{F \text{ could be a pdf.}}$

ii. $\boxed{F \text{ could be a cdf.}}$

iii. F is definitely not a pdf.

iv. F is definitely not a cdf.

e. [2 points] $\int_{-\infty}^{x} F(t)\ dt = \frac{1}{1+e^{-x}}$ for all x.

i. $\boxed{F \text{ could be a pdf.}}$

ii. F could be a cdf.

iii. F is definitely not a pdf.

iv. $\boxed{F \text{ is definitely not a cdf.}}$

6. [7 points] Determine if the following series converges or diverges using the **Limit Comparison Test**, and circle the corresponding word. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests you use.

$$\sum_{n=1}^{\infty} \frac{\sqrt{2n^3 - n^2 + 1}}{5n^2 + n - 4}$$

Circle one: **Converges** $\boxed{\textbf{Diverges}}$

*Justification (using the **Limit Comparison Test**):*

Solution: We aim to compare with the series $\sum \frac{1}{n^{1/2}}$. We see that

$$\lim_{n\to\infty} \frac{\frac{\sqrt{2n^3-n^2+1}}{5n^2+n-4}}{\frac{1}{n^{1/2}}} = \lim_{n\to\infty} \frac{n^{1/2}\sqrt{2n^3 - n^2 + 1}}{5n^2 + n - 4}$$
$$= \lim_{n\to\infty} \frac{\sqrt{2}n^2}{5n^2}$$
$$= \frac{\sqrt{2}}{5},$$

and so the Limit Comparison Test does apply.
By the p-test for series (with $p = \frac{1}{2}$), $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges. Therefore, by the Limit Comparison Test,

$$\sum_{n=1}^{\infty} \frac{\sqrt{2n^3 - n^2 + 1}}{5n^2 + n - 4}$$

also diverges.

7. [10 points] Determine if the following series absolutely converges, conditionally converges, or diverges, and circle the corresponding word. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests you use.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sin\left(\frac{1}{n}\right) + \sqrt{n^2+1}}$$

Hint: the function $\sin(\frac{1}{x}) + \sqrt{x^2+1}$ *is a monotone increasing function for* $x \geq 2$.

Circle one: **Converges Absolutely** $\boxed{\textbf{Converges Conditionally}}$ **Diverges**

Justification:

Solution: Note that the series is alternating. Let $a_n = \dfrac{1}{\sin\left(\frac{1}{n}\right) + \sqrt{n^2+1}}$. Then for all n, $0 < a_{n+1} < a_n$ (using the hint), and we also have $\lim\limits_{n\to\infty} a_n = 0$. Therefore, by the Alternating Series Test, $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\sin\left(\frac{1}{n}\right) + \sqrt{n^2+1}}$ converges.

Now consider the series $\displaystyle\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{\sin\left(\frac{1}{n}\right) + \sqrt{n^2+1}}\right| = \sum_{n=2}^{\infty} \frac{1}{\sin\left(\frac{1}{n}\right) + \sqrt{n^2+1}}$.

Note that for $n \geq 2$, we have $\dfrac{1}{\sin\left(\frac{1}{n}\right) + \sqrt{n^2+1}} \geq \dfrac{1}{n + \sqrt{2n^2}} = \dfrac{1}{(1+\sqrt{2})n}$.

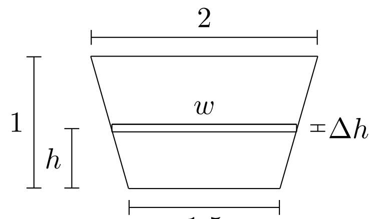
By the p-test, with $p = 1$, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(1+\sqrt{2})n}$ diverges, and so by the (Direct) Comparison Test, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sin\left(\frac{1}{n}\right) + \sqrt{n^2+1}}$ diverges.

Therefore, $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\sin\left(\frac{1}{n}\right) + \sqrt{n^2+1}}$ converges conditionally.

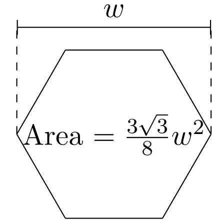
8. [11 points] Rachel is relaxing in her custom hexagonal hot tub when the gemstone from her engagement ring, a cherished gift from William, slips off and sinks to the bottom. Desperate to retrieve it, she decides to pump all the foaming water out over the top edge of the tub.

As viewed from the side, the hot tub is in the shape of a trapezoid, as depicted on the left below, and its horizontal cross-sections are regular hexagons. The diameter of the hot tub is 2 meters at the top rim and 1.5 meters at the bottom, and the tub has a height of 1 meter.

In this problem, you may use the fact that the area of a regular hexagon with diameter w is given by $A = \frac{3\sqrt{3}}{8}w^2$, and that the acceleration due to gravity is $g = 9.8$ m/s^2.



Side view of hot tub



Hexagonal cross-section with diameter w

a. [2 points] Let w be the diameter of a thin horizontal slice of water at a height h meters above the bottom of the tub. Find a formula for w in terms of h.

Solution: Observe that w is a linear function of h with slope 0.5 and $w(0) = 1.5$:

$$w = 1.5 + 0.5h$$

Answer: $w =$ $1.5 + 0.5h$

b. [5 points] Suppose the density of foaming water at h meters above the bottom of the tank, in kg/m^3, is given by the function $\rho(h)$. Write an expression which approximates the **weight** of a horizontal slice of the foaming water which is h meters above the bottom of the tank, and which has small thickness Δh. Your expression may include the letters h, w, and/or ρ, but should not contain any integrals. Include units.

Solution: The volume of a horizontal slice of the foaming water at height h is given by

$$\frac{3\sqrt{3}}{8}(1.5 + 0.5h)^2\Delta h.$$

The approximate weight of the slice is obtained by multiplying by the density, and multiplying the resultant mass by the acceleration due to gravity, giving

$$9.8\rho(h)\frac{3\sqrt{3}}{8}(1.5 + 0.5h)^2\Delta h.$$

Answer: $9.8\rho(h)\frac{3\sqrt{3}}{8}(1.5 + 0.5h)^2\Delta h$ **Units:** N

c. [4 points] The tub is initially filled with water to a height of 0.8m above the **bottom** of the tub. Write an expression involving an integral for the total work needed to pump all of the water over the top edge of the hot tub. Your expression may include the letters h, w, and/or ρ. Include units.

Solution: The approximate work required to lift a slice of water at height h over the top of the hot tub is given by multiplying its weight by the distance traveled:

$$9.8\rho(h)\frac{3\sqrt{3}}{8}(1.5+0.5h)^2(1-h)\Delta h$$

Integrate from $h=0$ to $h=0.8$ to get the total work done:

$$\text{Total Work} = \int_0^{0.8} 9.8\rho(h)\frac{3\sqrt{3}}{8}(1.5+0.5h)^2(1-h)\ dh$$

Answer: $\int_0^{0.8} 9.8\rho(h)\frac{3\sqrt{3}}{8}(1.5+0.5h)^2(1-h)\ dh$ **Units:** J

9. [12 points] Suppose that a_n and b_n are sequences with the following properties:

- $\displaystyle\sum_{n=1}^{\infty}(-1)^n a_n$ converges.
- $\displaystyle\frac{1}{n^2} \leq b_n \leq \frac{1}{\sqrt{n}}$ for all $n \geq 1$.

For the following questions, determine if the statement is ALWAYS true, SOMETIMES true, or NEVER true, and circle the corresponding answer. Justification is not required.

a. [2 points] The sequence a_n diverges.

Circle one: **ALWAYS** **SOMETIMES** $\boxed{\textbf{NEVER}}$

b. [2 points] The series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Circle one: **ALWAYS** $\boxed{\textbf{SOMETIMES}}$ **NEVER**

c. [2 points] The series $\displaystyle\sum_{n=1}^{\infty} (a_n)^2$ converges.

Circle one: **ALWAYS** $\boxed{\textbf{SOMETIMES}}$ **NEVER**

d. [2 points] The series $\displaystyle\sum_{n=1}^{\infty} b_n$ converges.

Circle one: **ALWAYS** $\boxed{\textbf{SOMETIMES}}$ **NEVER**

e. [2 points] The series $\displaystyle\sum_{n=1}^{\infty} (b_n)^3$ converges.

Circle one: $\boxed{\textbf{ALWAYS}}$ **SOMETIMES** **NEVER**

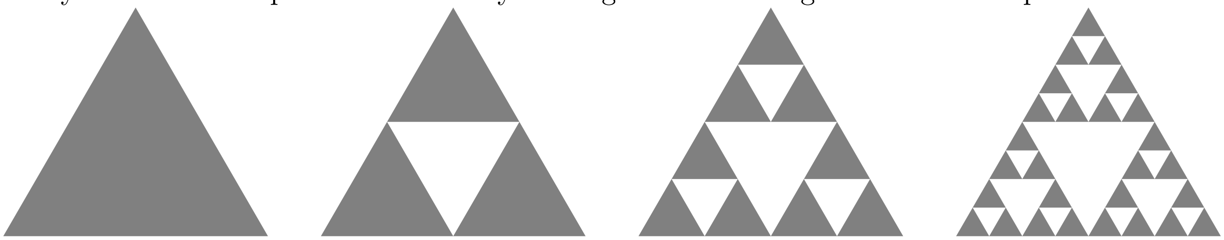
f. [2 points] The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{b_n}$ converges.

Circle one: **ALWAYS** **SOMETIMES** $\boxed{\textbf{NEVER}}$

10. [9 points] Molly constructs a pattern using paper in the shape of an equilateral triangle of side length 1.

- **Step 1:** Molly divides the triangle into 4 congruent triangles, and removes the triangle in the middle.
- **Step 2:** For each of the remaining triangles, Molly repeats the previous step.

Molly continues this process indefinitely. A diagram illustrating the first few steps is shown below.



$n = 0$ $n = 1$ $n = 2$ $n = 3$

The area of an equilateral triangle with side length a is given by $\frac{\sqrt{3}}{4}a^2$.

a. [4 points] For $n \geq 1$, let A_n be the **total** area of all the triangles Molly has **removed** after n steps. For example, $A_1 = \frac{\sqrt{3}}{4}\left(\frac{1}{2}\right)^2$. Find A_2 and A_3. You do not need to simplify your expressions.

Solution: We see that

$$A_2 = A_1 + 3 \cdot \frac{\sqrt{3}}{4}\left(\frac{1}{4}\right)^2 = \frac{\sqrt{3}}{4}\left(\frac{1}{2}\right)^2 + \frac{3}{4}\left(\frac{\sqrt{3}}{4}\left(\frac{1}{2}\right)^2\right) = \frac{\sqrt{3}}{16}\left(1 + \frac{3}{4}\right)$$

and

$$A_3 = A_2 + 3^2 \cdot \frac{\sqrt{3}}{4}\left(\frac{1}{2^3}\right)^2 = \frac{\sqrt{3}}{4}\left(\frac{1}{2}\right)^2 + \frac{3}{4}\left(\frac{\sqrt{3}}{4}\left(\frac{1}{2}\right)^2\right) + \left(\frac{3}{4}\right)^2\left(\frac{\sqrt{3}}{4}\left(\frac{1}{2}\right)^2\right)$$

$$= \frac{\sqrt{3}}{16}\left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2\right)$$

Answer: $A_2 = \frac{\sqrt{3}}{16}\left(1 + \frac{3}{4}\right)$

Answer: $A_3 = \frac{\sqrt{3}}{16}\left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2\right)$

b. [4 points] Find a **closed-form** expression for A_n. Closed form means your answer should not include ellipses or sigma notation, and should NOT be recursive.

Solution: Following the pattern in part **a.**, we have

$$A_n = \frac{\sqrt{3}}{16}\left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots + \left(\frac{3}{4}\right)^n\right) = \frac{\sqrt{3}}{16}\sum_{k=0}^{n-1}\left(\frac{3}{4}\right)^k$$

Using the formula for the sum of a finite geometric series:

$$A_n = \frac{\sqrt{3}}{16}\sum_{k=0}^{n-1}\left(\frac{3}{4}\right)^k = \frac{\sqrt{3}}{16}\frac{1-\left(\frac{3}{4}\right)^n}{1-\frac{3}{4}}$$

Answer: $A_n =$ $\frac{\sqrt{3}}{16}\left(\frac{1-\left(\frac{3}{4}\right)^n}{1-\frac{3}{4}}\right)$

c. [1 point] Find the limit $\lim_{n\to\infty} A_n$.

Answer: $\lim_{n\to\infty} A_n =$ $\frac{\sqrt{3}}{4}$

11. [12 points]

a. [6 points] For the following questions, determine if the statement is ALWAYS true, SOMETIMES true, or NEVER true, and circle the corresponding answer. Justification is not required.

(i) Suppose that the sequence a_n is defined by the recurrence relation $a_{n+1} = -\frac{1}{a_n}$ for all $n \geq 1$, where $a_1 > 0$. Then the sequence a_n is monotone.

Circle one: **ALWAYS** **SOMETIMES** $\boxed{\textbf{NEVER}}$

(ii) Suppose that b_n is a sequence and that $S_n = \sum_{k=1}^{n} b_k$ is the corresponding sequence of partial sums. Suppose that $\sum_{k=1}^{\infty} b_k$ converges. Then the sequence S_n is bounded.

Circle one: $\boxed{\textbf{ALWAYS}}$ **SOMETIMES** **NEVER**

(iii) Suppose that c is a positive constant. Then the geometric series $1 + 2c + 4c^2 + 8c^3 + \ldots$ converges to $\frac{1}{1-2c}$.

Circle one: **ALWAYS** $\boxed{\textbf{SOMETIMES}}$ **NEVER**

b. [6 points] For each of the following sequences, defined for $n \geq 1$, decide whether the sequence is monotone increasing, monotone decreasing, or not monotone, and whether it is bounded or unbounded. Circle your answers. No justification is required.

(i) $r_n = e^{n+1} - e^n$

*Circle **all** which apply:*

$\boxed{\textbf{Monotone Increasing}}$ **Monotone Decreasing** **Not Monotone**

Bounded $\boxed{\textbf{Unbounded}}$

(ii) $s_n = \int_0^{2^n} e^{-x^2}\, dx$

*Circle **all** which apply:*

$\boxed{\textbf{Monotone Increasing}}$ **Monotone Decreasing** **Not Monotone**

$\boxed{\textbf{Bounded}}$ **Unbounded**

(iii) $t_n = \sum_{k=1}^{n} \frac{(-1)^k}{k}$

*Circle **all** which apply:*

Monotone Increasing **Monotone Decreasing** $\boxed{\textbf{Not Monotone}}$

$\boxed{\textbf{Bounded}}$ **Unbounded**