Math 116 — Final Exam — April 27, 2026

EXAM SOLUTIONS

1. Please neatly write your 8-digit UMID number, your initials, your instructor's first and/or last name, and your section number in the spaces provided.
2. This exam has 15 pages including this cover.
3. There are 11 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is <u>not</u>.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	12	
2	12	
3	7	
4	9	
5	10	
6	5	

Problem	Points	Score
7	9	
8	7	
9	10	
10	10	
11	9	
Total	100	

1. [12 points] Lara is working on homework while sitting outside when she notices a beetle crawling on her graph paper. The beetle's position t seconds after she starts watching is given by

$$\begin{cases} x(t) = (t-3)(t-5) \\ y(t) = 5\sin\left(\dfrac{\pi t}{4}\right) \end{cases}$$

where $(0,0)$ is the origin, and the units on the x and y axes are centimeters (cm).

a. [4 points] Does the beetle ever pass through the origin? Justify your answer.

Solution: For the beetle to pass through the origin, it must satisfy $x = 0$ and $y = 0$. We see that $x = 0$ only when $t = 3$ or $t = 5$. We have $y(3) = 5\sin\left(\frac{3\pi}{4}\right) = \frac{5}{\sqrt{2}}$ and $y(5) = 5\sin\left(\frac{5\pi}{4}\right) = -\frac{5}{\sqrt{2}}$, and so we do not ever have $y = 0$ when $x = 0$. Therefore the beetle does not pass through the origin.

b. [4 points] Write an expression involving one or more integrals that gives the total distance, in cm, traveled by the beetle during the first three seconds that Lara is watching. **Do not** evaluate your integral(s). Show your work.

Solution: We have $x'(t) = 2t - 8$, and $y'(t) = \frac{5\pi}{4}\cos\left(\frac{\pi t}{4}\right)$, so during the first three seconds, the beetle travels a total of

$$\int_0^3 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_0^3 \sqrt{(2t-8)^2 + \left(\frac{5\pi}{4}\cos\left(\frac{\pi t}{4}\right)\right)^2}\, dt \quad \text{cm.}$$

Answer: $\int_0^3 \sqrt{(2t-8)^2 + \left(\frac{5\pi}{4}\cos\left(\frac{\pi t}{4}\right)\right)^2}\, dt$

c. [4 points] What is the equation of the tangent line to the beetle's path at $t = 3$? Give your answer using Cartesian coordinates (expressing y in terms of x only).

Solution: Since $x(3) = 0$, and $y(3) = \frac{5}{\sqrt{2}}$ (as we saw in part **a.**), we know the tangent line must pass through the point $\left(0, \frac{5}{\sqrt{2}}\right)$.

Using our formulas for $x'(t)$ and $y'(t)$ from part **b.**, we see that $x'(3) = -2$, and $y'(3) = \frac{5\pi}{4}\cos\left(\frac{3\pi}{4}\right) = -\frac{25}{4\sqrt{2}}$, and therefore the slope of the tangent line is $\frac{y'(3)}{x'(3)} = \frac{25}{8\sqrt{2}}$.

Therefore the tangent line is described by $y = \frac{25}{8\sqrt{2}}x + \frac{5}{\sqrt{2}}$.

Answer: $y =$ $\frac{25}{8\sqrt{2}}x + \frac{5}{\sqrt{2}}$

2. [12 points] The Taylor series centered at $x = -2$ for a function $f(x)$ is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{((n+1)!)^2}{2^n(2n+1)!}(x+2)^{3n+1}$$

a. [7 points] Determine the **radius** of convergence of the Taylor series above. Show all of your work. You do **not** need to find the interval of convergence.

Solution: We use the ratio test to find the radius of convergence. First, we form

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(((n+1)+1)!)^2|(x+2)^{3(n+1)+1}|}{2^{n+1}(2(n+1)+1)!} \cdot \frac{2^n(2n+1)!}{((n+1)!)^2|(x+2)^{3n+1}|} \\ &= \frac{2^n}{2^{n+1}} \cdot \frac{((n+2)!)^2}{((n+1)!)^2} \cdot \frac{(2n+1)!}{(2n+3)!} \cdot \frac{|x+2|^{3n+4}}{|x+2|^{3n+1}} \\ &= \frac{1}{2} \cdot (n+2)^2 \cdot \frac{1}{(2n+3)(2n+2)} \cdot |x+2|^3\end{aligned}$$

Now, we evaluate the limit:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{1}{2} \cdot \frac{(n+2)^2}{(2n+3)(2n+2)} \cdot |x+2|^3 \\ &= \lim_{n\to\infty} \frac{n^2}{8n^2}|x+2|^3 \\ &= \frac{1}{8}|x+2|^3\end{aligned}$$

The ratio test tells us that the Taylor series converges when this value is less than 1, i.e., $\frac{1}{8}|x+2|^3 < 1$. Rearranging the inequality, we find that $|x+2|^3 < 8$, which implies that the radius of convergence is 2.

Answer: 2

b. [5 points] Find $f^{(60)}(-2)$ and $f^{(61)}(-2)$. You **do not** need to simplify your answers.

Solution: Observe that the power of $(x+2)$ is always $3n+1$. If we set $3n+1=60$, we get $n=59/3$. Since there is no integer n that produces a degree 60 term, the coefficient of $(x+2)^{60}$ will be 0. Hence $f^{(60)}(-2)=0$.
On the other hand, $(x+2)^{61}$ appears when $3n+1=61$, i.e. when $n=20$. Thus

$$\frac{f^{(61)}(-2)}{61!} = \frac{((20+1)!)^2}{2^{20}(2\cdot 20+1)!}$$

Rearranging and simplifying, we get

$$f^{(61)}(-2) = \frac{61!(21!)^2}{2^{20}41!}$$

Answer: $f^{(60)}(-2) =$ 0 and $f^{(61)}(-2) =$ $\dfrac{61!(21!)^2}{2^{20}41!}$

3. [7 points] After the successful splashdown of the Artemis II mission, the Orion spacecraft is hoisted from the surface of Pacific Ocean by a crane onto the recovery ship. Before the crane begins lifting, the combined weight of the Orion capsule and the trapped seawater is $12,000$ pounds. As soon as the hoisting begins, the trapped seawater drains out of the capsule at a constant rate of 20 pounds per second. The capsule is lifted until it reaches the deck of the ship, 50 feet above the surface of the water. The crane lifts the capsule at a constant vertical rate of 2 feet per second.

a. [3 points] Let $C(h)$ be the combined weight of the capsule and the trapped water, in pounds, when the capsule is h feet above the surface of the water. Write an expression for $C(h)$.

Solution: Observe that

$$\frac{dC}{dh} = \frac{dC}{dt} \Big/ \frac{dh}{dt} = -20/2 = -10$$

Hence, $C(h)$ is a linear function of h with slope -10. Given the weight of the capsule at sea level ($h = 0$) is $12,000$ pounds, we obtain

$$C(h) = 12000 - 10h.$$

Answer: $C(h) =$ $12000 - 10h$

b. [4 points] Write an expression involving one or more integrals that represents the total work required to move the Orion capsule from the surface of the water to the deck of the ship using the crane. Your answer should not involve the letter C. **Do not** evaluate your integral. Include units.

Solution: Suppose the Orion capsule has already been lifted to a height h feet above the sea surface. The work required to lift it an additional Δh feet is given by:

$$C(h)\,\Delta h = (12000 - 10h)\,\Delta h.$$

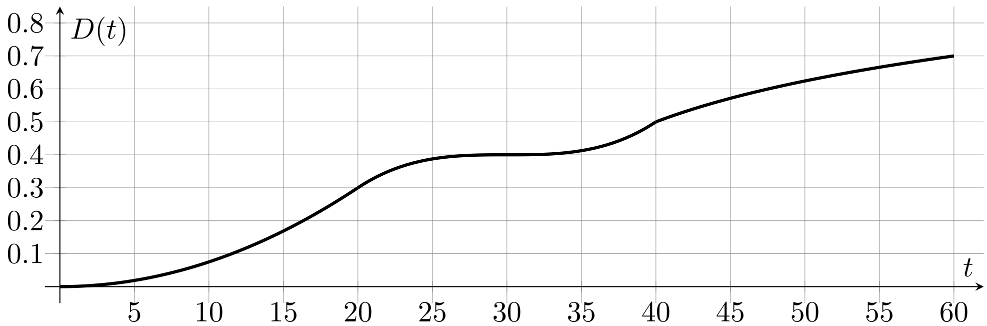
Therefore, the total work required to lift the capsule from the surface of the pacific ocean to the deck of the ship is given by

$$W = \int_0^{50} (12000 - 10h)\ \mathrm{d}h.$$

Answer: $\int_0^{50} (12000 - 10h)\ \mathrm{d}h$ **Units:** ft $\cdot$ lb

4. [9 points] A local basketball team measures their performance using different statistics.

a. [6 points] Dusty the basketball coach keeps track of t, the number of seconds into a game that his team scores their first basket. The **cumulative distribution function** (cdf), $D(t)$, for this quantity is defined for all real numbers. A **portion of the graph** for $D(t)$ is shown below.



For each of the following parts, find the exact value of the given quantity, or **if there is not enough information** to determine the quantity, write "NEI". No justification is required.

(i) $D(-2)$

Answer: 0

(ii) The probability that the team's first basket is scored within the first 20 seconds of the game

Answer: 0.3

(iii) The probability that it takes the team more than 60 seconds to score their first basket

Answer: 0.3

(iv) The probability that it takes the team between 30 and 40 seconds to score their first basket

Answer: 0.1

(v) The median number of seconds until the team scores their first basket

Answer: 40

(vi) The mean number of seconds until the team scores their first basket

Answer: NEI

b. [3 points] One of the players, Yaxel, tracks the number of minutes m that he spends on the court in each game. The **probability density function** (pdf) for this quantity is $Y(m)$. Complete the following sentence to give a practical interpretation of the equation $Y(30) = 0.14$.

The probability that Yaxel spends between 29 and 31 minutes on the court is ...

Solution: ...about 28%

5. [10 points] A power series centered at $x = -1$ is given by

$$\sum_{n=1}^{\infty} \frac{n}{3^n(n^2+1)}(x+1)^n.$$

The radius of convergence of this power series is 3 (do **not** show this). Find the **interval** of convergence of this power series. Show all your work, including full justification for series behavior.

Solution: Since we know the radius of convergence, we just need to test the behavior at the endpoints, which are $-1-3=-4$, and $-1+3=2$.

At $x=-4$, the series is

$$\sum_{n=1}^{\infty} \frac{n}{3^n(n^2+1)}(-4+1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$$

To determine the behavior of this, we use the Alternating Series Test. Set $a_n = \dfrac{n}{n^2+1}$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{n}{n^2} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

and for all $n \geq 1$,

$$0 < a_{n+1} < a_n,$$

so by the Alternating Series test, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ converges.
Therefore $x=-4$ is included in the interval of convergence.

At $x=2$, the series is

$$\sum_{n=1}^{\infty} \frac{n}{3^n(n^2+1)}(2+1)^n = \sum_{n=1}^{\infty} \frac{n}{n^2+1}.$$

We can show that this series diverges using either the Direct Comparison Test or the Limit Comparison Test; here, we'll use the Direct Comparison Test.
For $n \geq 1$, $\dfrac{n}{n^2+1} > \dfrac{n}{n^2+n^2} = \dfrac{1}{2n}$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges, by the p-test, with $p=1$. Therefore, by the (Direct) Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges. This tells us that $x=2$ is not included in the interval of convergence.
Therefore, the interval of convergence is $[-4,2)$.

Interval of convergence: $[-4,2)$

6. [5 points] For the function $f(x) = \cos(\sin(x))$, find $P_2(x)$, the Taylor polynomial of degree 2 for $f(x)$ centered about $x = \pi$. Show all your work.

Solution: To find the Taylor polynomial of degree 2 centered at $x = \pi$, we need to evaluate the function and its first two derivatives at $x = \pi$. First, we compute the derivatives:

$$\begin{aligned} f(x) &= \cos(\sin(x)) \\ f'(x) &= -\sin(\sin(x))\cos(x) \\ f''(x) &= -\cos(\sin(x))\cos^2(x) + \sin(\sin(x))\sin(x) \end{aligned}$$

Now, we evaluate these at $x = \pi$:

$$\begin{aligned} f(\pi) &= \cos(\sin(\pi)) = \cos(0) = 1 \\ f'(\pi) &= -\sin(\sin(\pi))\cos(\pi) = -\sin(0)(-1) = 0 \\ f''(\pi) &= -\cos(\sin(\pi))\cos^2(\pi) + \sin(\sin(\pi))\sin(\pi) = -\cos(0)(-1)^2 + 0 = -1 \end{aligned}$$

Using the formula for the Taylor polynomial $P_2(x) = f(\pi) + f'(\pi)(x-\pi) + \frac{f''(\pi)}{2!}(x-\pi)^2$, we get:

$$P_2(x) = 1 + 0(x-\pi) + \frac{-1}{2!}(x-\pi)^2 = 1 - \frac{1}{2}(x-\pi)^2$$

Answer: $P_2(x) =$ $1 - \frac{1}{2}(x-\pi)^2$

7. [9 points] Consider the function $g(x) = 3x\sin\left(\frac{x^3}{2}\right)$.

a. [5 points] Give the first three nonzero terms of the Taylor series for $g(x)$ centered about $x = 0$. Show all your work.

Solution: Using the Taylor series expansion of $\sin(y)$ with $y = \frac{x^3}{2}$ about $x = 0$, we obtain

$$\sin\left(\frac{x^3}{2}\right) = \frac{x^3}{2} - \frac{x^9}{3!2^3} + \frac{x^{15}}{5!2^5} + \cdots$$

Therefore, the Taylor series of $g(x)$ about $x = 0$ is

$$g(x) = 3x\sin\left(\frac{x^3}{2}\right) = 3x\left(\frac{x^3}{2} - \frac{x^9}{3!2^3} + \frac{x^{15}}{5!2^5} + \cdots\right) = \frac{3x^4}{2} - \frac{3x^{10}}{3!2^3} + \frac{3x^{16}}{5!2^5} + \cdots$$

Answer: $\frac{3x^4}{2} - \frac{3x^{10}}{3!2^3} + \frac{3x^{16}}{5!2^5} + \cdots$

b. [4 points] The function $g(x)$ has a continuous antiderivative, $G(x)$, with a Taylor series that converges for all x. Given that $G(0) = 2$, find the first four non-zero terms of the Taylor series for $G(x)$ centered about $x = 0$. Show all your work.

Solution: The Second Fundamental Theorem of Calculus implies

$$G(x) = \int_0^x g(t)\,\mathrm{d}t + 2.$$

Using the Taylor series expansion of $g(t)$ from part **a**, and integrating term by term, we obtain the Taylor series expansion of $G(x)$ about $x = 0$:

$$\begin{aligned} G(x) &= 2 + \int_0^x \frac{3t^4}{2}\,\mathrm{d}t - \int_0^x \frac{3t^{10}}{3!2^3}\,\mathrm{d}t + \int_0^x \frac{3t^{16}}{5!2^5}\,\mathrm{d}t + \cdots \\ &= 2 + \frac{3x^5}{2\cdot 5} - \frac{3x^{11}}{3!\cdot 2^3\cdot 11} + \frac{3x^{17}}{5!\cdot 2^5\cdot 17} + \cdots \end{aligned}$$

Answer: $2 + \frac{3x^5}{10} - \frac{3x^{11}}{3!\cdot 2^3\cdot 11} + \frac{3x^{17}}{5!\cdot 2^5\cdot 17} + \cdots$

8. [7 points] Consider functions f and g that satisfy all of the following:

- $f(x)$ is defined, continuous, and positive for all $x > 1$.
- $\lim_{x\to 1^+} f(x) = \infty$ (so $f(x)$ has a vertical asymptote at $x = 1$).
- $g(x)$ is defined and differentiable for all real numbers, and $g'(x)$ is continuous.
- $F(x) = \dfrac{g(x)}{x^2-1}$ is an antiderivative of $f(x)$ for all $x > 1$,
- Some values of the functions g and g' appear in the below table.

x	1	5
$g(x)$	0	-3
$g'(x)$	-8	2

Compute the value of the following improper integral if it converges. If it does not converge, use a **direct computation** of the integral to show its divergence. Be sure to show your full computation, and be sure to use **proper notation**.

$$\int_1^5 f(x)\,\mathrm{d}x$$

Circle one: **Diverges** **Converges to** $\dfrac{31}{8}$

Solution: We start by rewriting this improper integral as a limit, and then use the First Fundamental Theorem of Calculus:

$$\begin{aligned}\int_1^5 f(x)\,\mathrm{d}x &= \lim_{b\to 1^+}\int_b^5 f(x)\,\mathrm{d}x\\ &= \lim_{b\to 1^+} F(5) - F(b)\\ &= \lim_{b\to 1^+} \frac{g(5)}{24} - \frac{g(b)}{b^2-1}\\ &= -\frac{3}{24} - \lim_{b\to 1^+}\frac{g(b)}{b^2-1}.\end{aligned}$$

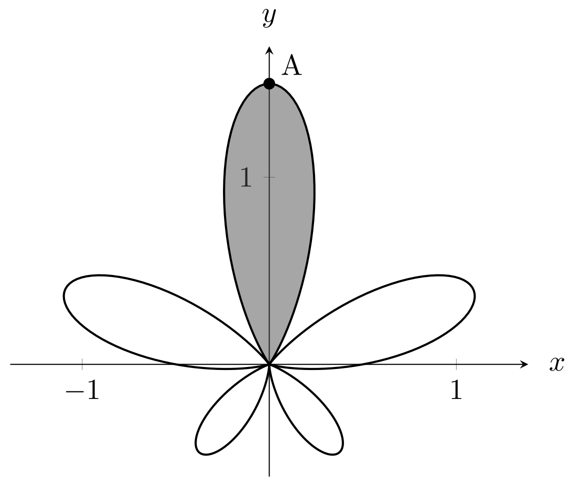
As $\lim_{b\to 1^+} g(b) = \lim_{b\to 1^+} b^2 - 1 = 0$, we try to use L'Hôpital's Rule. We obtain:

$$\begin{aligned}\int_1^5 f(x)\,\mathrm{d}x &= -\frac{1}{8} - \lim_{b\to 1^+}\frac{g'(b)}{2b}\\ &= -\frac{1}{8} + \frac{8}{2} = 4 - \frac{1}{8} = \frac{31}{8}.\end{aligned}$$

Therefore, $\int_1^5 f(x)\,\mathrm{d}x$ converges to $\dfrac{31}{8}$.

9. [10 points]

As part of a new logo design for his varsity team, Jake wants to recreate the shape of a buckeye leaf, the symbol of his home state. He models the leaf's outline, shown in the figure to the right, using the polar equation $r = \left(\frac{1}{2} + \frac{1}{2}\sin(5\theta)\right)\left(1 + \frac{1}{2}\sin(\theta)\right)$, where the leaf is traced out once as θ ranges from 0 to 2π.



a. [4 points] Find all the values of θ in the interval $[0, 2\pi)$ for which the curve passes through the origin.

Solution: To find the values of θ in the interval $[0, 2\pi)$ where the curve passes through the origin, we set $r = 0$. From the factorization $r = \left(\frac{1}{2} + \frac{1}{2}\sin(5\theta)\right)\left(1 + \frac{1}{2}\sin(\theta)\right)$, we observe that $r = 0$ when

$$\frac{1}{2} + \frac{1}{2}\sin(5\theta) = 0 \text{ or equivalently, } \sin(5\theta) = -1$$

The equation $\sin(5\theta) = -1$ has five solutions in the interval $[0, 2\pi)$: $\theta = \dfrac{3\pi}{10} + \dfrac{2n\pi}{5}$, where $n = 0, 1, 2, 3, 4$.

Answer: $\theta =$ $\dfrac{3\pi}{10}, \dfrac{7\pi}{10}, \dfrac{11\pi}{10}, \dfrac{3\pi}{2}, \dfrac{19\pi}{10}$

b. [2 points] The point A, labeled above, is the tip of the buckeye leaf. Write A in polar coordinates (r, θ), with $0 \le \theta < 2\pi$.

Solution: The point A sits on the y-axis. Hence, $\theta = \frac{\pi}{2}$. Evaluate r at $\theta = \frac{\pi}{2}$, we obtain

$$r\left(\frac{\pi}{2}\right) = \left(\frac{1}{2} + \frac{1}{2}\sin\left(5\frac{\pi}{2}\right)\right)\left(1 + \frac{1}{2}\sin\left(\frac{\pi}{2}\right)\right) = \frac{3}{2}$$

Answer: $A : (r, \theta) =$ $\left(\dfrac{3}{2}, \dfrac{\pi}{2}\right)$

c. [4 points] Jake wants to paint the largest leaflet green, shown as the shaded region in the figure. Write an expression involving one or more integrals that represents the area of this leaflet. **Do not** evaluate your expression.

Solution: From part **a**, we determine that the shaded region is traced for θ in the interval $\left(\frac{3\pi}{10}, \frac{7\pi}{10}\right)$. Using the formula for the area, the total area of the shaded region is given by

$$\int_{\frac{3\pi}{10}}^{\frac{7\pi}{10}} \frac{1}{2}\left(\left(\frac{1}{2}+\frac{1}{2}\sin(5\theta)\right)\left(1+\frac{1}{2}\sin(\theta)\right)\right)^2 \, d\theta.$$

Answer: $\int_{\frac{3\pi}{10}}^{\frac{7\pi}{10}} \frac{1}{2}\left(\left(\frac{1}{2}+\frac{1}{2}\sin(5\theta)\right)\left(1+\frac{1}{2}\sin(\theta)\right)\right)^2 \, \mathbf{d}\theta.$

10. [10 points] For the following questions, determine if the statement is ALWAYS true, SOMETIMES true, or NEVER true, and circle the corresponding answer. Justification is not required.

a. [2 points] Suppose that $f(x)$ is a continuous function with a continuous antiderivative $F(x)$ which satisfies $F(1) = 5$. Suppose that $\int_1^4 f(t)\, dt = 6$, and let $H(x) = \int_4^x f(t)\, dt$. Then $F(2026) - H(2026) = 11$.

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

b. [2 points] Suppose that $g(x)$ is a continuous function which is concave up on the interval $[2, 5]$, and that the TRAP(5) estimate to $\int_2^5 g(x)\, dx$ is 11. Then the average value of $g(x)$ on the interval $[2, 5]$ is at least 4.

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

c. [2 points] Suppose that k is an integer. Then the point $(x, y) = (-1, 0)$ in the xy-plane can be described as $(r, \theta) = (1, 2k\pi)$ in polar coordinates.

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

d. [2 points] Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges. Then the power series $\sum_{n=1}^{\infty} a_n x^n$ converges for all x.

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

e. [2 points] Suppose k is a positive constant and that the improper integral $\int_e^{\infty} \frac{1}{x \ln(x)^k}\, dx$ converges. Then the series $\sum_{n=1}^{\infty} \frac{1}{n^k}$ also converges.

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

11. [9 points] Consider the following initial value problem:

$$y''(t) - 16y(t) = 0$$

with initial conditions $y(0) = 1$ and $y'(0) = 4$. As in the team homework, suppose the solution $y(t)$ to this initial value problem takes the form of a power series centered at $t = 0$:

$$y(t) = \sum_{n=0}^{\infty} c_n t^n = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n + \cdots$$

so that we have:

$$y'(t) = c_1 + 2c_2 t + 3c_3 t^2 + \ldots + (n+1)c_{n+1} t^n + \ldots,$$
$$y''(t) = 2c_2 + 3 \cdot 2c_3 t + 4 \cdot 3c_4 t^2 + \ldots + (n+2)(n+1)c_{n+2} t^n + \ldots$$

a. [2 points] Use the initial conditions $y(0) = 1$ and $y'(0) = 4$ to find c_0 and c_1.

Answer: $c_0 =$ 1 and $c_1 =$ 4

b. [2 points] Substituting $y(t)$ and $y''(t)$ into the differential equation yields the following equation, which must hold for all values of t:

$$\sum_{n=0}^{\infty} \left((n+2)(n+1)c_{n+2} - 16c_n\right) t^n = 0.$$

Use this to find a recurrence relation for c_{n+2} in terms of c_n which applies for all $n \geq 0$.

Solution: We need all coefficients to be equal to 0, and so we need $(n+2)(n+1)c_{n+2} - 16c_n = 0$ for all n. Rearranging, we see $c_{n+2} = \dfrac{16}{(n+1)(n+2)} c_n$

Answer: $c_{n+2} = \dfrac{16}{(n+1)(n+2)} c_n$

c. [5 points] We can use the recurrence relation to show that for all $n \geq 0$,

$$c_{2n} = \frac{4^{2n}}{(2n)!} \quad \text{and} \quad c_{2n+1} = \frac{4^{2n+1}}{(2n+1)!}.$$

Use these coefficients to express $y(t)$ as a power series centered around 0 using sigma ($\sum$) notation. Then, use the known Taylor series formula to find a closed-form expression for $y(t)$.

Solution: For each n, we have $c_n = \dfrac{4^n}{n!}$, and so we have $y(t) = \displaystyle\sum_{n=0}^{\infty} \frac{4^n}{n!} t^n$.
Using our Known Taylor series, we see that we can rewrite $y(t) = e^{4t}$.

Using sigma notation, $y(t) =$ $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{n!} t^n$.

In closed form, $y(t) =$ e^{4t}.

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \le 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$

Select Values of Trigonometric Functions:

θ	$\sin\theta$	$\cos\theta$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$