8. (10 points) Let f be a continuous, positive function for $x \geq 1$.

(a) Define what it means to say that $\int_1^\infty f(x)\,dx$ converges.
Answer:

$\int_1^\infty f(x)dx$ converges if $\lim_{b\to\infty} \int_1^b f(x)dx$ is a finite number.

(b) If f from part (a) is such that $\int_1^\infty f(x)\,dx$ converges and if g is another continuous positive function for $x \geq 1$ that satisfies

$$g(x) \leq 5f(x) + \frac{3}{x^2}$$

then is it necessarily true that $\int_1^\infty g(x)\,dx$ converges? (Explain why or why not.)
Answer:
The fact that $\int_1^\infty f(x)dx$ converges tells us that $\int_1^\infty 5f(x)dx$ also converges. We also know from class that $\int_1^\infty \frac{1}{x^2}dx$ converges, so $\int_1^\infty \frac{3}{x^2}dx$ converges as well. Therefore, the sum $\int_1^\infty (5f(x)+\frac{3}{x^2})dx$ converges. Since $\int_1^\infty g(x)dx \leq \int_1^\infty (5f(x)+\frac{3}{x^2})dx$, we can conclude by the comparision test that $\int_1^\infty g(x)dx$ converges.