8. (10 points) Let $f$ be a continuous, positive function for $x \geq 1$.

(a) Define what it means to say that $\int_{1}^{\infty} f(x) \, dx$ converges.

Answer:

$\int_{1}^{\infty} f(x) \, dx$ converges if $\lim_{b \to \infty} \int_{1}^{b} f(x) \, dx$ is a finite number.

(b) If $f$ from part (a) is such that $\int_{1}^{\infty} f(x) \, dx$ converges and if $g$ is another continuous positive function for $x \geq 1$ that satisfies

$$g(x) \leq 5f(x) + \frac{3}{x^2}$$

then is it necessarily true that $\int_{1}^{\infty} g(x) \, dx$ converges? (Explain why or why not.)

Answer:

The fact that $\int_{1}^{\infty} f(x) \, dx$ converges tells us that $\int_{1}^{\infty} 5f(x) \, dx$ also converges. We also know from class that $\int_{1}^{\infty} \frac{3}{x^2} \, dx$ converges, so $\int_{1}^{\infty} \frac{3}{x^2} \, dx$ converges as well. Therefore, the sum $\int_{1}^{\infty} (5f(x) + \frac{3}{x^2}) \, dx$ converges. Since $\int_{1}^{\infty} g(x) \, dx \leq \int_{1}^{\infty} (5f(x) + \frac{3}{x^2}) \, dx$, we can conclude by the comparison test that $\int_{1}^{\infty} g(x) \, dx$ converges.