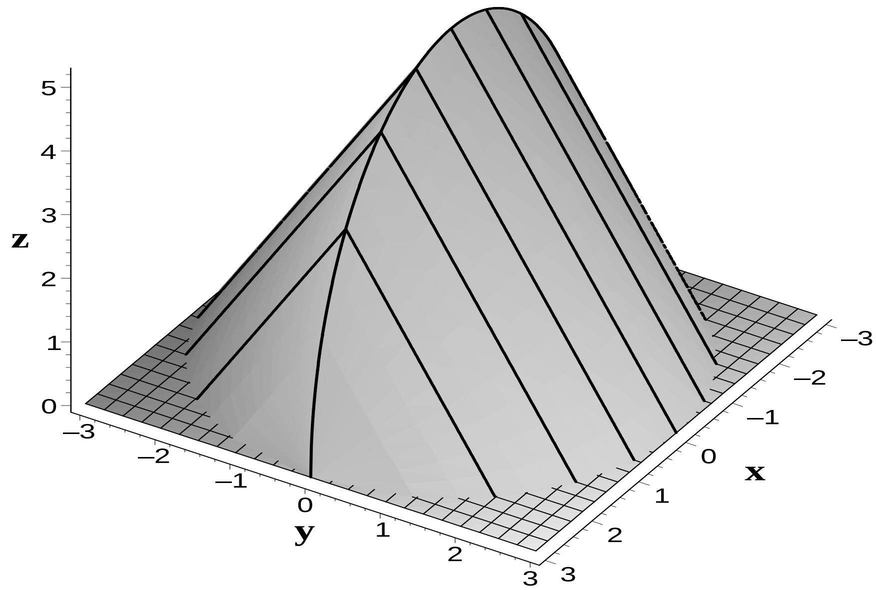
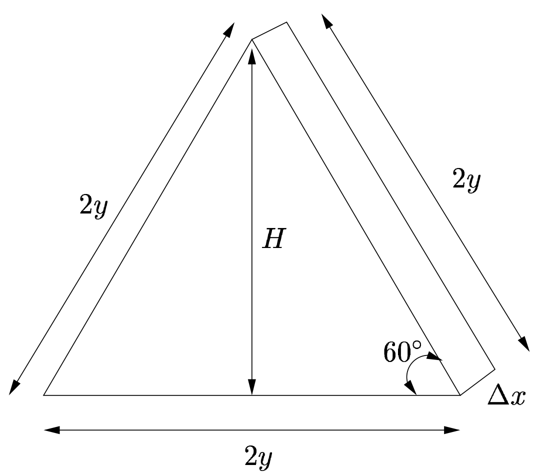
9. (12 points) It's a beautiful sunny day and you are at the beach. You manage to build the most spectacular sand castle ever. Unforunately, fate is cruel and a rogue wave hits the beach and washes over your sandcastle. But, fate also has a kinder side and it leaves you a shapely mound of sand as pictured below. The mound has as a base the interior of the circle $x^2 + y^2 = 9$ in the x-y plane and has cross sections by planes perpendicular to the x-axis given by equilateral triangles with one side in the x-y plane.



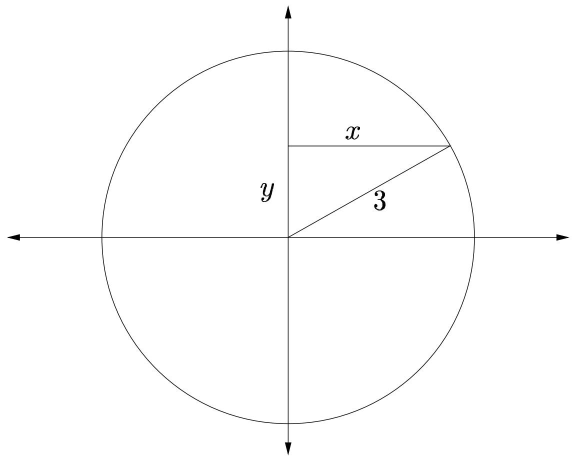
(a) Sketch and label the dimensions of a typical slice of the sand mound perpendicular to the x-axis for $-3 < x < 3$. What is the volume of this slice in terms of x?



The volume of this slice is $V_{\text{slice}} = \frac{1}{2}\, 2yH\Delta x = yH\Delta x$. In order to put this in terms of x, we need to express H and y in terms of x. We can use some trigonometry to write $H = 2y \sin 60 = \sqrt{3}y$. To write y in terms of x, we use the fact that we know the base satisfies the equation $x^2 + y^2 = 9$.

From this figure we see that $y = \sqrt{9 - x^2}$. So our formula for the volume of a slice becomes

$$\begin{aligned} V_{\text{slice}} &= 2y\, H\, \Delta x \\ &= \sqrt{3}(\sqrt{9 - x^2})\,(\sqrt{9 - x^2})\,\Delta x \\ &= \sqrt{3}\,(9 - x^2)\Delta x \end{aligned}$$



(b) Write a Riemann sum and then a definite integral representing the volume of the sand pile.

Answer:
The volume of the sand pile can be approximated by adding up all the slices of volume found in part (a). This gives the Riemann sum :

$$\begin{aligned} V_{\text{sand pile}} &= \sum V_{\text{slice}} \\ &= \sum \sqrt{3}\,(9 - x^2)\,\Delta x \end{aligned}$$

Now let $\Delta x \to 0$, so the Riemann sum becomes a definite integral. The volume of the slice is then given by

$$V_{\text{sand pile}} = \sqrt{3}\int_{-3}^{3} (9 - x^2)\,dx$$

(c) Find the exact volume of the solid. If you can't compute the volume exactly, give the most accurate approximation you can and explain how you found it.

Answer:
This integral is an elementary integral to evaluate, involving only power functions.

$$\begin{aligned} V_{\text{sand pile}} &= \sqrt{3}\int_{-3}^{3} (9 - x^2)\,dx \\ &= \sqrt{3}\left(9x - \frac{1}{3}x^3\right)\Big|_{-3}^{3} \\ &= 36\sqrt{3} \end{aligned}$$