9. (12 points) It's a beautiful sunny day and you are at the beach. You manage to build the most spectacular sand castle ever. Unfortunately, fate is cruel and a rogue wave hits the beach and washes over your sandcastle. But, fate also has a kinder side and it leaves you a shapely mound of sand as pictured below. The mound has as a base the interior of the circle $x^2 + y^2 = 9$ in the $x$-$y$ plane and has cross sections by planes perpendicular to the $x$-axis given by equilateral triangles with one side in the $x$-$y$ plane.

(a) Sketch and label the dimensions of a typical slice of the sand mound perpendicular to the $x$-axis for $-3 < x < 3$. What is the volume of this slice in terms of $x$?

The volume of this slice is $V_{\text{slice}} = \frac{1}{2}2yH\Delta x = yH\Delta x$. In order to put this in terms of $x$, we need to express $H$ and $y$ in terms of $x$. We can use some trigonometry to write $H = 2y\sin 60 = \sqrt{3}y$. To write $y$ in terms of $x$, we use the fact that we know the base satisfies the equation $x^2 + y^2 = 9$.

From this figure we see that $y = \sqrt{9 - x^2}$. So our formula for the volume of a slice becomes

\[
V_{\text{slice}} = 2yH\Delta x = \sqrt{3}(\sqrt{9 - x^2}) (\sqrt{9 - x^2}) \Delta x = \sqrt{3}(9 - x^2)\Delta x
\]
(b) Write a Riemann sum and then a definite integral representing the volume of the sand pile.

Answer:
The volume of the sand pile can be approximated by adding up all the slices of volume found in part (a). This gives the Riemann sum:

\[ V_{\text{sand pile}} = \sum V_{\text{slice}} = \sum \sqrt{3} (9 - x^2) \Delta x \]

Now let \( \Delta x \to 0 \), so the Riemann sum becomes a definite integral. The volume of the slice is then given by

\[ V_{\text{sand pile}} = \sqrt{3} \int_{-3}^{3} (9 - x^2) \, dx \]

(c) Find the exact volume of the solid. If you can’t compute the volume exactly, give the most accurate approximation you can and explain how you found it.

Answer:
This integral is an elementary integral to evaluate, involving only power functions.

\[ V_{\text{sand pile}} = \sqrt{3} \int_{-3}^{3} (9 - x^2) \, dx = \sqrt{3} \left[ 9x - \frac{1}{3} x^3 \right]_{-3}^{3} = 36 \sqrt{3} \]