4. (10 points) Give the definition of the improper integral $\int_1^\infty \frac{1}{x^{3/2}}\,dx$. Then use your definition to evaluate the integral if it converges, or else show it diverges.

The given integral is improper because of the upper limit being infinite. To handle this, we define the improper integral as a limit of definite integrals:

$$\int_1^\infty \frac{1}{x^{3/2}}dx \;=\; \lim_{b\to+\infty}\int_1^b \frac{1}{x^{3/2}}dx \;=\; \lim_{b\to+\infty}\left[\frac{-2}{x^{1/2}}\right]_1^b \;=\; -2\lim_{b\to+\infty}\frac{1}{b^{1/2}} + 2 \;=\; 2\,.$$

Therefore the improper integral **converges** *and has the value 2.*

5. (12 points). Evaluate the integrals, given that $f(x)$ is a continuous function for $0 \le x \le 6$ with the following properties:

$$f(0) = 2,\quad f(2) = 3,\quad f(4) = -1,\quad f(6) = 5;\qquad\qquad f'(0) = 1,\quad f'(2) = 4;$$
$$\int_0^2 f(x)\,dx = 3,\quad \int_2^4 f(x)\,dx = 1,\quad \int_4^6 f(x)\,dx = 6.$$

(a) $\int_0^2 xf'(x)\,dx \;=\; \mathbf{3}.$

By parts using $u = x$ and $v' = f'$, so $u' = 1$ and $v = f$, we find:

$$\int_0^2 xf'(x)dx \;=\; \big[xf(x)\big]_0^2 - \int_0^2 f(x)dx \;=\; 2\times 3 - 0\times 2 - 3 \;=\; 3\,.$$

(b) $\int_2^4 f'(x)\big(2+3f(x)\big)\,dx \;=\; \mathbf{-20}.$

Break up the integral into two pieces. The first one is done using the FTC directly, while the second one is handled by noting $f'(x)f(x) = \frac{1}{2}\big(f^2(x)\big)'$.

$$\begin{aligned}\int_2^4 f'(x)\big(2+3f(x)\big)dx &= 2\int_2^4 f'(x)dx + \frac{3}{2}\int_2^4 \big(f^2(x)\big)'dx\\ &= 2\,(f(4)-f(2)) + \frac{3}{2}\big(f(4)^2 - f(2)^2\big) \;=\; -20\,.\end{aligned}$$

(c) $\int_0^2 f(3x)\,dx \;=\; \mathbf{\frac{10}{3}}.$

By substitution, setting $u = 3x$, so $dx = du/3$ and the new limits of integration 0 and 6, we find:

$$\int_0^2 f(3x)dx \;=\; \frac{1}{3}\int_0^6 f(u)du \;=\; \frac{1}{3}\left(\int_0^2 f(u)du + \int_2^4 f(u)du + \int_4^6 f(u)du\right) \;=\; \frac{3+1+6}{3} \;=\; \frac{10}{3}.$$