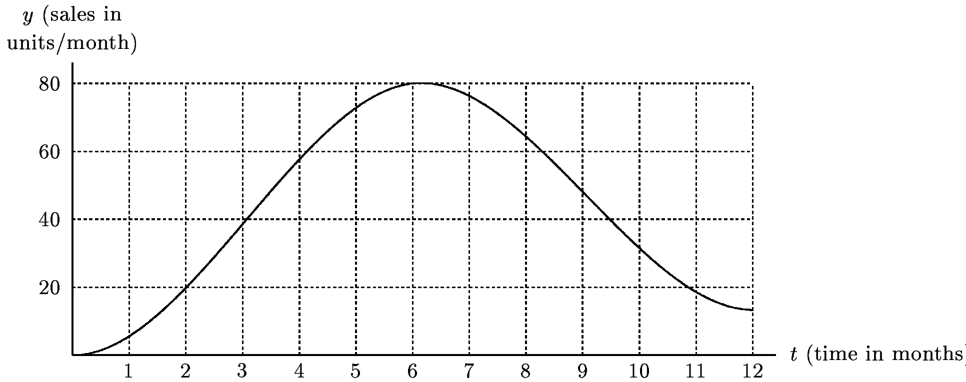
**7.** (18 points) Aberister is a small company which sells clothing items for youngsters in North America. When the new designs are released on January 1, 2005, the company projects sales $y(t)$ of shirts (in units of 1,000 items per month) for the year 2005 to be as shown in the figure. As the figure shows, sales are expected to increase until the summer when most "target" customers will have made their purchases, and then decline for the remainder of the year.




**(a)** Write an expression for $A(t)$, the projected monthly average number of units which will have been sold in the first $t$ months of the year.

$$A(t) \;=\; \frac{1}{t}\int_0^t y(s)\,ds \quad \textit{units.}$$

**(b)** Why is the value of $A(t)$ always less than 80, for any value of $0 < t \le 12$?

*The function $A(t)$ is the average value of $y(t)$ and $y$ is always smaller than 80, so its average is as well. Analyticially, this is because*

$$A(t) \;=\; \frac{1}{t}\int_0^t y(s)\,ds \;\le\; \frac{1}{t}\int_0^t 80\,ds \;=\; 80\,.$$

**(c)** Approximate the value of $A(t)$ at $t = 3$. Explain how you arrived at your approximation.

*The area of each box under the graph of $y(t)$ has the interpretation as 20 units of sales or 20,000 clothing items. Interpreting $A(3)$ as the area under the graph of $A$ from $t = 0$ to $t = 3$, we see this is approximately equal to two and a quarter boxes, or 45 sales units. That is,*

$$A(3) \;=\; \frac{1}{3}\int_0^3 y(t)\,dt \;\approx\; \frac{45}{3} \;=\; \mathbf{15} \quad \textbf{\textit{units.}}$$

*(The actual value from the function whose graph is shown in the figure is about 14.46 units)*

ANSWER: $A(3) \approx$ 15 units

Solution continued on next page.

Solution continued from previous page.

Assuming the price of the shirts remains constant throughout the year, Aberister will maximize its profit on the 2005 designs by launching its next collection, the holiday seasons designs, when $A(t)$ is maximum.

**(d)** Suppose that at time $t_{max}$ the value of $A$ is maximum. What is the relationship between the values of $y$ and $A$ at time $t_{max}$? Explain.

*At a maximum value of $A(t)$, we must have $A'(t_{max}) = 0$. By the quotient rule and then the FTC, we have*

$$A'(t) = \frac{t\frac{d}{dt}\int_0^t y(s)\,ds - \int_0^t y(s)\,ds}{t^2} = \frac{ty(t) - \int_0^t y(s)\,ds}{t^2}.$$

*The equation $A'(t_{max}) = 0$ implies that the numerator of this fraction vanishes. Dividing the numerator by $t_{max}$ then gives*

$$y(t_{max}) = \frac{1}{t_{max}}\int_0^{t_{max}} y(s)\,ds = A(t_{max}).$$

**(e)** Sketch a graph of the function $A(t)$ on the above figure (previous page). Be sure to show where $A(t)$ is increasing and decreasing. Use your graph to estimate the time of the year when Aberister should launch its new collection.

*The function $A(t)$ will be increasing so long as $y(t) > A(t)$ and decreasing if $y(t) < A(t)$. Therefore, its graph must cross the graph of $y(t)$ at the point where $A(t)$ is a maximum. We also saw this in part* **(d)**, $y(t_{max}) = A(t_{max})$.

*The geometric interpretation of the average function $A(t)$ is that it is the height where, to the left of $t$, the area of the region below the graph of $y$ and above the height is equal to the area above the graph of $y$ and below the height. So, to find the maximum of $A(t)$ we just have to find the point $(t, y(t))$ where the height $y(t)$ has this property. That is, if we look at the horizontal line of height $y(t)$, the area to the left of $t$, below the graph of $y$, and above the horizontal line through $(t, y(t))$ is equal to the area to the left of $t$, above the graph of $y$, and below the horizontal line. This occurs at approximately* $\mathbf{t} = \mathbf{9}$, *where the height is a little less than 50 (actual height is 49). See graphs below.*