8. (14 points) A team of biologists seeking to develop alternate solutions to the use of pesticides proposes the following experiment. A plant infested by a large colony of aphids (small insects), is placed in a container originally saturated in dioxygen \( \text{O}_2 \). They hope that the parasites can be suffocated by the \( \text{CO}_2 \) produced by their natural activities (e.g. breathing) and those of the plant (e.g. photosynthesis). Make the simplifying assumptions, which are approximations, that:

(i) The container is sealed; no molecules enter or leave.

(ii) The sum of the volumes of \( \text{O}_2 \) and \( \text{CO}_2 \) in the container is constant and equal to 117 deciliters (dL); i.e. 11.7 liters.

(iii) At the start of the experiment \( (t = 0) \), there is no \( \text{CO}_2 \) in the container.

The light in the room is adjusted so as to mimick a “perfect” 24-hour period. The parasites produce \( \text{CO}_2 \) at a constant rate of 7.5 dL per hour. When the light is turned on, the plant absorbs \( \text{CO}_2 \) and releases \( \text{O}_2 \). When the light is turned off, it does the opposite; i.e. uses \( \text{O}_2 \) and produces \( \text{CO}_2 \). The rates of release in the container of \( \text{CO}_2 \) (in dL per hour) for the parasites, \( a(t) \), and the plant, \( p(t) \), are plotted below. Note that negative rates correspond to absorption.

(a) By approximately how much does the volume \( V(t) \) of \( \text{CO}_2 \) in the container change in the small time interval between \( t \) and \( t + \Delta t \)? Express your answer in terms of \( \Delta t \), and of the rates \( p(t) \) and \( a(t) \).

\[
\text{The production of } \text{CO}_2 \text{ in the tank comes from both the aphids and the plant. So the total rate at which } \text{CO}_2 \text{ is released is given by } a(t) + p(t). \text{ Thus, in the small time interval from } t \text{ to } t + \Delta t, \text{ the (small) change in the volume of } \text{CO}_2 \text{ is :} \]

\[
\Delta V = (a(t) + p(t)) \Delta t.
\]
(b) Write an integral that gives the volume \( V(12) \) of \( CO_2 \) in the container after 12 hours. Explain why your integral gives the value of \( V(12) \). You will probably want to use the answer from (a) in your explanation.

\[ V(12) = \int_0^{12} (a(t) + p(t)) \, dt. \]

(c) What is the value of \( V(12) \)? Explain how you obtained your answer.

The integral in (b) represents the sum of the area under the graph of \( a(t) \) and the area under the graph of \( p(t) \), between \( t = 0 \) and \( t = 12 \). To find its value, it suffices to count the “boxes” on the chart. The scale on the figure shows that each box represents a volume of 9 dL of \( CO_2 \). Counting the boxes gives:

\[ V(12) = (10 - 8) \times 9 = 18 \text{ deciliters}. \]

(d) When the container becomes saturated in \( CO_2 \), i.e. no \( O_2 \) remains, the parasites suffocate to death, and the experiment is stopped. Decide whether that will happen within the 24-hour period; and, if you think it will, estimate this time.

We are told there are 117 dL of \( O_2 \) available at \( t = 0 \). All aphids will thus die if the the volume of \( CO_2 \) in the container reaches the value 117. To decide whether that will happen before the end of the experiment, i.e. before \( t = 24 \) hours, we evaluate \( V(24) \) by counting the boxes as was done in (c). Doing so gives:

\[ V(24) = (20 - 8 + 4) \times 9 = 144 > 117 \text{ deciliters}. \]

Accordingly, the aphids will indeed suffocate to death before the experiment ends. Again counting the boxes, we find that will happen after 21 hours. Indeed, at \( t = 21 \) the total volume of \( CO_2 \) in the container is:

\[ V(21) = (17.5 - 8 + 3.5) \times 9 = 117 \text{ deciliters}. \]