6. (13 points) Let a and b be positive numbers. The region bounded by the positive y-axis, the positive x-axis, the vertical line $x = b$ and the curve $y = e^{-ax}$ is revolved about the x-axis.

(a) (6 pts.) Find the volume of the resulting solid. (Yes, your answer will involve a and b.)

Have,

$$\text{Volume of a slice } (\mathrm{V}_{\text{slice}}) \simeq \pi r^2 \Delta x, \qquad \text{where } r = e^{-ax}.$$

Therefore,

$$\mathrm{V}_{\text{slice}} \simeq \pi (e^{-ax})^2 \Delta x = \pi e^{-2ax} \Delta x.$$

and so

$$\begin{aligned}\text{Total Volume } &= \int_0^b \pi e^{-2ax}\, dx = \left.\frac{\pi e^{-2ax}}{-2a}\right|_0^b \\ &= \frac{\pi e^{-2ab}}{-2a} - \frac{\pi}{-2a} \\ &= \frac{\pi}{2a}(1 - e^{-2ab}).\end{aligned}$$

(b) (7 pts.) Suppose we let $b \to \infty$, creating a solid with an infinitely long neck. Does this solid have finite volume? If so, find it (showing step-by-step work.) If not, explain why not.

$$V = \lim_{b\to\infty} \frac{\pi}{2a}(1 - e^{-2ab}) = \frac{\pi}{2a}\ ,$$

since $e^{-2ab} \to 0$ as $b \to \infty$, (because $a > 0$).

So,

$$V = \frac{\pi}{2a} \text{ is finite.}$$