4. [8 points] Use the fact that $\int_0^\infty e^{-x}\sin(x)\,dx = \frac{1}{2}$ to find $\int_0^\infty e^{-x}\cos(x)\,dx$.

Solution:
We use integration by parts on $\int_0^\infty e^{-x}\cos(x)\,dx$ with $u = e^{-x}$, so that $u' = -e^{-x}$ and $v' = \cos(x)$, so that $v = \sin(x)$. Then

$$\begin{aligned}\int_0^\infty e^{-x}\cos(x)\,dx &= \lim_{b\to\infty}\left(e^{-x}\sin(x)\Big|_0^b + \int_0^b e^{-x}\sin(x)\,dx\right)\\ &= \lim_{b\to\infty}\left(e^{-b}\sin(b)\right) + \int_0^\infty e^{-x}\sin(x)\,dx\\ &= 0 + \frac{1}{2}.\end{aligned}$$

Thus $\int_0^\infty e^{-x}\cos(x)\,dx = \frac{1}{2}$.
Alternate solution: with $u = \sin(x)$ and $v' = e^{-x}$, we have $u' = \cos(x)$ and $v = -e^{-x}$, so that $\int_0^\infty e^{-x}\sin(x)\,dx = \lim_{b\to\infty}(-\cos(b)\,e^{-b} + 1) - \int_0^\infty e^{-x}\sin(x)\,dx = 1 - \frac{1}{2} = \frac{1}{2}$.
Second alternate solution: Use integration by parts twice to solve for answer without using the given $\int_0^\infty e^{-x}\sin(x)\,dx = \frac{1}{2}$. Note that this doesn't completely follow the instructions, and therefore cannot receive full credit.

5. [8 points] Let $F(x) = \int_0^{x^2(x-1)} g(t)\,dt$, where $g(t)$ is always positive. For what values of x is $F(x)$ increasing? For what values is it decreasing?

Solution:
$F(x)$ is increasing when $F'(x) > 0$ and decreasing when $F'(x) < 0$. Taking the derivative, $F'(x) = \frac{d}{dx}\int_0^{x^2(x-1)} g(t)\,dt = (2x(x-1)+x^2)\cdot g(x^2(x-1))$. Because g is always positive, the sign of this expression is determined by the first term, $2x(x-1)+x^2 = 3x^2 - 2x$. We can see when this is positive and negative by graphing it, or by factoring. Factoring, $3x^2 - 2x = x(3x-2)$, which is negative for $0 < x < \frac{2}{3}$, and positive for $x < 0$ and $x > \frac{2}{3}$. Thus $F(x)$ is increasing for $x < 0$ and $x > \frac{2}{3}$, and decreasing for $0 < x < \frac{2}{3}$.