8. [12 points] In class, Chris' calculus professor is well known to cover material at a rate $m(t) = \frac{1}{12(t-20)^{2/3}}$ textbook sections/minute, where t is the time in minutes since the start of class.

(a) [2 of 12 points] What is the meaning of the integral $\int_0^{80} m(t)\,dt$ (include units in your explanation)?

Solution:
The integral is the area under the rate $m(t)$ between $t = 0$ and $t = 80$, which, by the Fundamental Theorem of Calculus is the total number of sections that Chris' professor covers in the 80 minute (90 less ten) class period.

(b) [4 of 12 points] How many sections would you estimate the professor covers in the first minute of class? In the 20th minute? Why?

Solution:
We note that $m(0) = \frac{1}{12(20)^{2/3}} = 0.0113$ sections/minute. Thus we might guess that the professor might cover approximately 0.0113 sections-worth of material in the first minute. If we repeat this calculation for the 20th minute, we might guess that the number of sections covered is $(1\,\text{minute})(m(19)) = \frac{1}{12(1)^{2/3}} = \frac{1}{12}$ sections. Note, however, that $m(20)$ is undefined—therefore, it appears that the professor is speaking at an infinite rate at about $t = 20$, so that we might also wonder if an infinite number of words are spoken. We can verify this by completing part (c) of this problem.

(c) [6 of 12 points] Find exactly (that is, by hand) the value of $\int_0^{80} m(t)\,dt$.

Solution:
We note that because $m(t)$ is discontinuous at $t = 20$ this is an improper integral. We therefore evaluate

$$\begin{aligned}
\int_0^{80} \frac{1}{12(t-20)^{2/3}}\,dt &= \lim_{a\to 20^-} \int_0^{a} \frac{1}{12(t-20)^{2/3}}\,dt + \lim_{a\to 20^+} \int_a^{80} \frac{1}{12(t-20)^{2/3}}\,dt \\
&= \lim_{a\to 20^-} \frac{1}{4}(t-20)^{1/3}\Big|_0^a + \lim_{a\to 20^+} \frac{1}{4}(t-20)^{1/3}\Big|_a^{80} \\
&= \lim_{a\to 20^-} \frac{1}{4}\left((a-20)^{1/3} + (20)^{1/3}\right) + \lim_{a\to 20^+} \frac{1}{4}\left((60)^{1/3} - (a-20)^{1/3}\right) \\
&= \frac{1}{4}(20)^{1/3} + \frac{1}{4}(60)^{1/3} \text{ sections.}
\end{aligned}$$

Or, approximately 1.66 sections per class period. Because this integral is finite, it is clear that the amount of material covered in the 20th minute is, in fact, finite too.