9. [10 points] To improve their understanding of the material in their Calculus course, Alex and Chris have invented a set of statements about the material they have been studying. These statements are given below. For each statement, circle **true** (that is, the statement is always true), or **false** (it isn't), and give a one sentence explanation for your answer.

(a) [2 of 10 points] If a bounded continuous function $f(x)$ has the properties that $f(x) > \frac{1}{x}$ for $1 < x < 50$, $f(50) = \frac{1}{50}$, and $f(x) < \frac{1}{x}$ for $x > 50$, then $\int_1^\infty f(x)\,dx$ converges.

TRUE FALSE

Solution:
False. The behavior of $f(x)$ for $x \le 50$ doesn't matter for the convergence of $\int_1^\infty f(x)\,dx$, but because $\int_1^\infty \frac{1}{x}\,dx$ diverges, $f(x) < \frac{1}{x}$ for $x > 50$ only tells us that the area under $f(x)$ is guaranteed to be "less infinite" than that under $\frac{1}{x}$.

(b) [2 of 10 points] If a bounded continuous function $f(x)$ has the properties that $f(x) > \frac{1}{x^2}$ for $1 < x < 50$, $f(50) = \frac{1}{2500}$, and $f(x) < \frac{1}{x^2}$ for $x > 50$, then $\int_1^\infty f(x)\,dx$ converges.

TRUE FALSE

Solution:
True. Using the same logic as in (a), we know that the area under $f(x)$ is less than the area under $\frac{1}{x^2}$; thus, because $\int_1^\infty \frac{1}{x^2}\,dx$ converges, so too must $\int_1^\infty f(x)\,dx$. As a side note, this argument assumes that $f(x) \ge 0$, which should have been included in the conditions on $f(x)$ given in the problem.

(c) [2 of 10 points] Since the function $\frac{\sin(x)+2}{\sqrt{x}}$ is always less than $\frac{3}{\sqrt{x}}$ for $2 \le x < \infty$ and $\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$, we know that $\int_2^\infty \frac{\sin(x)+2}{\sqrt{x}}\,dx$ converges.

TRUE FALSE

Solution:
False. Again, while $\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$, $\int_2^\infty \frac{2}{\sqrt{x}}\,dx$ diverges, so we know only that $\int_2^\infty \frac{\sin(x)+1}{\sqrt{x}}\,dx$ is smaller than that infinite value, which provides no guarantee that it converges.

(d) [2 of 10 points] If $0 < \frac{1}{x} < g(x) < \frac{1}{x^2}$ for $0 < x < 1$, then the area between $g(x)$ and the x-axis for $0 < x < 1$ is guaranteed to be finite.

TRUE FALSE

Solution:
False. We know that both $\int_0^1 \frac{1}{x}\,dx$ and $\int_0^1 \frac{1}{x^2}\,dx$ diverge, so the integral of $g(x)$ must also diverge, and the area is correspondingly infinite.

(e) [2 of 10 points] Let $f(x) = \frac{1}{(x-1)^2}$. Then if $F(x) = \int_0^x f(t)\,dt$, we know that $F(0) = 0$ and that $F(2) = \int_0^2 \frac{1}{(t-1)^2}\,dt = -\frac{1}{t-1}\Big|_0^2 = -1 - 1 = -2$.

TRUE FALSE

Solution:
False. Because $\frac{1}{(x-1)^2}$ is undefined at $x = 1$, we need to evaluate $F(2)$ as an improper integral, which because the integrand looks like $(x-1)^{-2}$ near $x = 1$ will diverge.