2. [14 points] Consider the integral $\int_0^2 \frac{(\sin\sqrt{x})+1}{\sqrt{x}}\,dx$.

(a) [2 points of 14] Explain why this is an improper integral.

Solution:
We note that at $x = 0$ the denominator of the integrand is undefined, because of the factor of $\sqrt{x}$. Thus the integrand becomes infinite as $x \to 0$, and the integral is accordingly improper.

(b) [6 points of 14] Carefully show, using an appropriate comparison function (that is, without actually evaluating the integral), that this integral converges.

Solution:
We note that $0 < \sin\sqrt{x} + 1 < 2$ for $0 \le x \le 2$. Thus $\frac{\sin\sqrt{x}+1}{\sqrt{x}} < \frac{2}{\sqrt{x}}$ and $\int_0^2 \frac{\sin\sqrt{x}+1}{\sqrt{x}}\,dx < \int_0^2 \frac{2}{\sqrt{x}}\,dx = 2\int_0^2 \frac{1}{\sqrt{x}}\,dx = 2\int_0^1 \frac{1}{\sqrt{x}}\,dx + 2\int_1^2 \frac{1}{\sqrt{x}}\,dx$. We know that the last expression converges because it is the sum of an integral we know converges and a proper integral; as it is greater than our original integral, the original integral must similarly converge to a value less than the sum of the last two integrals.

(c) [6 points of 14] Carefully work out the actual value of the integral.

Solution:
To find the value of the integral we substitute $w = \sqrt{x}$, so that

$$\int_0^2 \frac{\sin\sqrt{x}+1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\int_a^2 \frac{\sin\sqrt{x}+1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\int_a^{\sqrt{2}} 2\sin w + 2\,dw$$
$$= \lim_{a\to 0^+}\left(-2\cos\sqrt{2} + 2\cos\sqrt{a} + 2\sqrt{2} - 2\sqrt{a}\right) = -2\cos\sqrt{2} + 2 + 2\sqrt{2}.$$