8. [12 points] The velocity of an object, with air resistance, may in some circumstances be given as

$$v(t) = \sqrt{\frac{g}{k}}\left(\frac{e^{2mt}}{e^{2mt}+1} - \frac{1}{e^{2mt}+1}\right),$$

where g is the acceleration due to gravity, k is a constant representing air resistance, and $m = \sqrt{gk}$.

(a) [2 points of 12] Write an expression for the distance D that the object falls in the first t_0 seconds.

Solution:
$D = \int_0^{t_0} v(t)\,dt = \sqrt{\frac{g}{k}}\int_0^{t_0}\left(\frac{e^{2mt}}{e^{2mt}+1} - \frac{1}{e^{2mt}+1}\right)dt.$

(b) [5 points of 12] Find the distance D *(note that half of this calculation is significantly harder than the rest; do not waste too much time on it if you get stuck).*

Solution:
We can integrate both parts of the integral with the substitutions $w = e^{2mt}$, so that $\frac{1}{2m}dw = e^{2mt}dt$, or, equivalently, $dt = \frac{1}{2me^{2mt}}dw = \frac{1}{2mw}dw$. Then

$$\begin{aligned}
D = \sqrt{\frac{g}{k}}\int_0^{t_0}\left(\frac{e^{2mt}}{e^{2mt}+1} - \frac{1}{e^{2mt}+1}\right)dt &= \frac{1}{2m}\sqrt{\frac{g}{k}}\left(\int_1^{e^{2mt_0}}\frac{1}{w+1}\,dw - \int_1^{e^{2mt_0}}\frac{1}{w(w+1)}\,dw\right)\\
&= \frac{1}{2m}\sqrt{\frac{g}{k}}\left(\ln(e^{2mt_0}+1) - \ln(2) - (\ln|w| - \ln|w+1|)\Big|_1^{e^{2mt_0}}\right)\\
&= \frac{1}{2m}\sqrt{\frac{g}{k}}\left(\ln(e^{2mt_0}+1) - \ln(2) - \ln(e^{2mt_0}) + \ln(e^{2mt_0}+1) - \ln(2)\right)\\
&= \sqrt{\frac{g}{k}}\left(\frac{\ln(e^{2mt_0}+1) - \ln(2)}{m} - t_0\right),
\end{aligned}$$

where we used partial fractions (or a table) to find the second integral.

(c) [5 points of 12] Suppose $\sqrt{g/k} = 10$ and $m = 1$. Note that in this case $v(3) = 9.95 \approx 10$. Use a geometric argument to show that the distance traveled between $t = 0$ and $t = 3$, $D(3)$, satisfies the inequality $15 < D(3) < 30$.

Solution:
A graph of $v(t)$ is shown to the right, along with the graph of $v = 10$ and $v = \frac{10}{3}t$ (for $0 \le t \le 3$). Clearly the actual distance traveled (the area under $v(t)$) is between the area under $v(t) = 10$ and that under $v(t) = \frac{10}{3}t$. These areas are, respectively, $d_1 = (10)(3) = 30$ and $d_2 = \frac{1}{2}(3)(10) = 15$. Thus $15 < D(3) < 30$.