8. [12 points] The velocity of an object, with air resistance, may in some circumstances be given as

\[ v(t) = \sqrt{\frac{g}{k}} \left( \frac{e^{2mt}}{e^{2mt} + 1} - \frac{1}{e^{2mt} + 1} \right) , \]

where \( g \) is the acceleration due to gravity, \( k \) is a constant representing air resistance, and \( m = \sqrt{gk} \).

(a) [2 points of 12] Write an expression for the distance \( D \) that the object falls in the first \( t_0 \) seconds.

**Solution:**

\[ D = \int_0^{t_0} v(t) \, dt = \sqrt{\frac{g}{k}} \int_0^{t_0} \left( \frac{e^{2mt}}{e^{2mt} + 1} - \frac{1}{e^{2mt} + 1} \right) \, dt. \]

(b) [5 points of 12] Find the distance \( D \) (note that half of this calculation is significantly harder than the rest; do not waste too much time on it if you get stuck).

**Solution:**

We can integrate both parts of the integral with the substitutions \( w = e^{2mt} \), so that \( \frac{1}{2m} \, dw = e^{2mt} \, dt \), or, equivalently, \( dt = \frac{1}{2me^{2mt}} \, dw = \frac{1}{2mw} \, dw \). Then

\[ D = \sqrt{\frac{g}{k}} \int_0^{t_0} \left( \frac{e^{2mt}}{e^{2mt} + 1} - \frac{1}{e^{2mt} + 1} \right) \, dt = \sqrt{\frac{g}{k}} \left( \int_1^{e^{2mt_0}} \frac{1}{w+1} \, dw - \int_1^{e^{2mt_0}} \frac{1}{w(w+1)} \, dw \right) \]

\[ = \frac{1}{2m} \sqrt{\frac{g}{k}} \left( \ln(e^{2mt_0} + 1) - \ln(2) - (\ln |w| - \ln |w+1|) \right) \]

\[ = \frac{1}{2m} \sqrt{\frac{g}{k}} \left( \ln(e^{2mt_0} + 1) - \ln(2) - \ln(e^{2mt_0}) + \ln(e^{2mt_0} + 1) - \ln(2) \right) \]

\[ = \sqrt{\frac{g}{k}} \left( \frac{\ln(e^{2mt_0} + 1) - \ln(2)}{m} - t_0 \right) , \]

where we used partial fractions (or a table) to find the second integral.

(c) [5 points of 12] Suppose \( \sqrt{g/k} = 10 \) and \( m = 1 \). Note that in this case \( v(3) = 9.95 \approx 10 \). Use a geometric argument to show that the distance traveled between \( t = 0 \) and \( t = 3 \), \( D(3) \), satisfies the inequality \( 15 < D(3) < 30 \).

**Solution:**

A graph of \( v(t) \) is shown to the right, along with the graph of \( v = 10 \) and \( v = \frac{10}{t} \) (for \( 0 \leq t \leq 3 \)). Clearly the actual distance traveled (the area under \( v(t) \)) is between the area under \( v(t) = 10 \) and that under \( v(t) = \frac{10}{t} \). These areas are, respectively, \( d_1 = (10)(3) = 30 \) and \( d_2 = \frac{10}{2}(3)(10) = 15 \). Thus \( 15 < D(3) < 30 \).