2. [12 points] Let $f(x)$ be a positive, continuous and differentiable, non-constant function. Let $F(x)$ be an antiderivative of f that passes through the origin. For each of the following, find all values of the constant a for which the statement is true. Include your work and/or a short explanation so that it is clear how you obtain your answers.

a. [4 points] $F(x) = \int_a^x f(t)\,dt$

Solution: This is true if $a = 0$. We know that $F(x)$ and $\int_a^x f(t)\,dt$ are both antiderivatives of $f(x)$. $F(x)$ passes through $(0, 0)$, while $\int_a^x f(t)\,dx$ passes through $(a, 0)$. Thus $a = 0$.

b. [4 points] $\int_0^a x\,f'(x)\,dx = f(a) - F(a)$.

Solution: This is true if $a = 1$. Using integration by parts, $\int_0^a x\,f'(x)\,dx = x\,f(x)\Big|_0^a - \int_0^a f(x)\,dx = a\,f(a) - F(a)$. For this to equal $f(a) - F(a)$, we must have $a = 1$.

c. [4 points] $\int 5\,f(\frac{x}{a})\,dx = F(\frac{x}{a}) + C$

Solution: This is true when $a = \frac{1}{5}$. Differentiating both sides of the equation, we get $5f(\frac{x}{a}) = \frac{1}{a}\,F'(\frac{x}{a}) = \frac{1}{a}\,f(\frac{x}{a})$. Thus a must equal $\frac{1}{5}$.
Alternately, integrating the left hand side, we have $\int 5\,f(\frac{x}{a})\,dx = 5aF(\frac{x}{a}) + C$. Thus for the two sides to be equal we must have $a = \frac{1}{5}$.