2. [12 points] Let \( f(x) \) be a positive, continuous and differentiable, non-constant function. Let \( F(x) \) be an antiderivative of \( f \) that passes through the origin. For each of the following, find all values of the constant \( a \) for which the statement is true. Include your work and/or a short explanation so that it is clear how you obtain your answers.

a. [4 points] \( F(x) = \int_a^x f(t) \, dt \)

Solution: This is true if \( a = 0 \). We know that \( F(x) \) and \( \int_a^x f(t) \, dt \) are both antiderivatives of \( f(x) \). \( F(x) \) passes through \((0, 0)\), while \( \int_a^x f(t) \, dt \) passes through \((a, 0)\). Thus \( a = 0 \).

b. [4 points] \( \int_0^a x f'(x) \, dx = f(a) - F(a) \)

Solution: This is true if \( a = 1 \). Using integration by parts, \( \int_0^a x f'(x) \, dx = x f(x) \bigg|_0^a - \int_0^a f(x) \, dx = a f(a) - F(a) \). For this to equal \( f(a) - F(a) \), we must have \( a = 1 \).

c. [4 points] \( \int 5 f\left( \frac{x}{a} \right) \, dx = F\left( \frac{x}{a} \right) + C \)

Solution: This is true when \( a = \frac{1}{5} \). Differentiating both sides of the equation, we get \( 5f\left( \frac{x}{a} \right) = \frac{1}{a} F'\left( \frac{x}{a} \right) = \frac{1}{a} f\left( \frac{x}{a} \right) \). Thus \( a \) must equal \( \frac{1}{5} \).

Alternately, integrating the left hand side, we have \( \int 5 f\left( \frac{x}{a} \right) \, dx = 5a F\left( \frac{x}{a} \right) + C \). Thus for the two sides to be equal we must have \( a = \frac{1}{5} \).