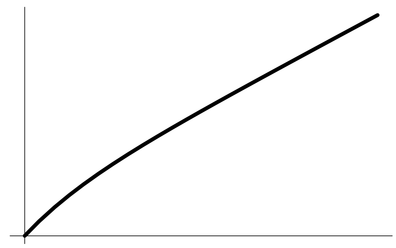
3. [14 points] An astute University of Michigan squirrel notes that the length of one strand of the web of a mathematically inclined spider is exactly $\int_0^6 \sqrt{2+2e^{-x}+e^{-2x}}\,dx$ cm, where x measures the horizontal distance from the wall of a campus building. The strand of web is shown in the figure to the right, below.



a. [7 points] Find an equation $y = f(x)$ that describes the shape of this strand of web.

Solution: We note that the squirrel is integrating to find the arclength of a curve. The arclength of a curve $y = f(x)$ is given by $\int_a^b \sqrt{1+(f'(x))^2}\,dx$, so we must have $(f'(x))^2 = 1 + 2e^{-x} + e^{-2x}$. Thus $f'(x) = \pm(1 + e^{-x})$. To match the graph shown we must take the positive sign, so $f(x) = x - e^{-x} + C$, for some constant C. Because the web starts at the origin ($(x, y) = (0, 0)$), C must be 1, so $y = x - e^{-x} + 1$.

b. [7 points] Estimate the length of the strand of web using MID(3). Is your estimate an over- or underestimate? How do you know?

Solution: We're estimating the integral $\int_0^6 \sqrt{2+2e^{-x}+e^{-2x}}\,dx$. The three intervals being considered are $0 \le x \le 2$, $2 \le x \le 4$, and $4 \le x \le 6$, and the midpoints of these intervals are $x = 1$, $x = 3$ and 5. Thus the midpoint approximation is

$$\text{MID}(3) = 2\left(\sqrt{2+2e^{-1}+e^{-2}} + \sqrt{2+2e^{-3}+e^{-6}} + \sqrt{2+2e^{-5}+e^{-10}}\right).$$

(Which is approximately 9.127 cm.) By graphing the function $g(x)$ we can see that it is concave up, so we know that the midpoint is an underestimate for the actual length.