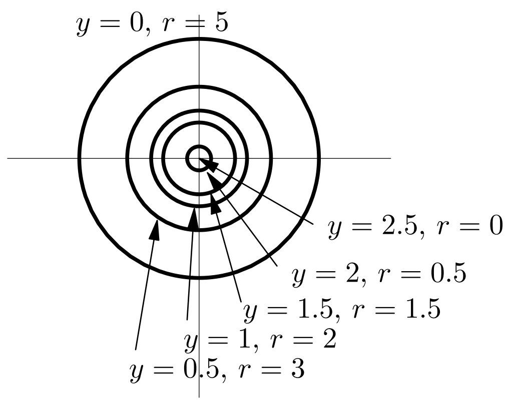
4. [16 points] A University of Michigan squirrel, in Peru for a study-abroad semester, discovers a singularly symmetric pond hidden high in the Andes mountains. The pond has perfectly circular horizontal cross-sections, and its radii r at different depths y are shown (in meters) in the figure to the right, below. As shown, the outer edge of the pond has a radius of 5 meters, and the pond gets deeper towards its center.

a. [5 points] Set up an integral that gives the total volume of the pond. Your integral may involve the radius (r) and/or depth (y) of the pond. Be sure it is clear how you obtain your answer.



Solution: The depth from the surface of the pond down to a "slice" of the pond is y. The radius of the pond at that depth is $r(y)$, so that the cross-sectional area of the slice at that depth is $\pi\,(r(y))^2$. Thus the volume of the slice is $\pi\,(r(y))^2\,\Delta y$, where Δy is the thickness of the slice. We can sum these up as $\Delta y \to 0$ by using an integral:

$$\text{Volume of pond} = \int_0^{2.5} \pi\,(r(y))^2\,dy.$$

(It is also possible to integrate vertical, circular shells with thickness Δr and height $y(r)$. This gives the integral $V = \int_0^5 2\pi\, r\, y(r)\, dr$.)

b. [5 points] Estimate the volume of the pond based on your work in (a).

Solution: Because the radii for the cross sections of the pond are given at intervals of $\Delta y = \frac{1}{2}$ m, we can use a left- or right-hand sum with this Δy.

$$\begin{aligned}\text{LEFT} &= \tfrac{\pi}{2}\left(r(0)^2 + r(1/2)^2 + r(1)^2 + r(3/2)^2 + r(2)^2\right)\\ &= \tfrac{\pi}{2}\left(5^2 + 3^2 + 2^2 + (3/2)^2 + (1/2)^2\right) = \tfrac{81\pi}{4} \approx 64\text{ m}^3\\ \text{RIGHT} &= \tfrac{\pi}{2}\left(3^2 + 2^2 + (3/2)^2 + (1/2)^2 + 0^2\right) = \tfrac{31\pi}{4} \approx 24\text{ m}^3.\end{aligned}$$

The trapezoid estimate is the average of these two, $\frac{56\pi}{4} \approx 44$ m^3.

c. [6 points] The pond is fed by a stream that is drying up as time goes on. If the stream delivers water to the pond at a rate of $r(t) = 60t\,e^{-t^2}$ m^3/year, does the pond ever fill? (Assume that the pond starts out empty when $t = 0$, and ignore other effects such as evaporation and rainfall.)

Solution: The total amount of water that will flow into the pond is given by $\int_0^\infty r(t)\,dt$, which is an improper integral. We find the value of this integral by taking the appropriate limit:

$$\int_0^\infty r(t)\,dt = \lim_{b\to\infty} \int_0^b 60t\,e^{-t^2}\,dt = \lim_{b\to\infty} -30\,e^{-t^2}\Big|_0^b = \lim_{b\to\infty} -30(e^{-b^2} - 1) = 30\text{ m}^3.$$

This is below the trapezoid estimate, so we would expect that the pond does not fill (though because it is between the left- and right-hand estimates, it might).