2. [12 points] A population of creatures is placed on a small preservation space. Ten creatures are initially placed on the preservation. The time it takes for a population to reach C creatures is given by

$$T(C) = \int_{10}^{C} \frac{20dx}{x(400-x)},$$

where T is measured in years after the creatures were first placed on the preservation.

a. [6 points] Find a function for $T(C)$ by analytically solving the integral given above. Be sure to show all appropriate work.

Solution: First we use partial fractions to rewrite the integrand.

$$\frac{20}{x(400-x)} = \frac{A}{x} + \frac{B}{400-x} = \frac{400A - Ax + Bx}{x(400-x)}$$

This gives us the conditions $A = B = \frac{1}{20} = 0.05$. We then have

$$\begin{aligned} T(C) &= \frac{1}{20}\int_{10}^{C} \frac{dx}{x} + \frac{1}{20}\int_{10}^{C} \frac{dx}{400-x} \\ &= \frac{1}{20}\ln|x|\big|_{10}^{C} - \frac{1}{20}\ln|400-x|\big|_{10}^{C} \\ &= \frac{1}{20}\ln|C| - \frac{1}{20}\ln|10| - \frac{1}{20}\ln|400-C| + \frac{1}{20}\ln|390| \\ &= \frac{1}{20}\ln|39| + \frac{1}{20}\ln\left|\frac{C}{400-C}\right| \end{aligned}$$

b. [2 points] How long does it take for the creatures to reach a population of 50? State your answer in a complete sentence and include units in your answer.

Solution:

$$T(C) = \frac{1}{20}\ln|39| + \frac{1}{20}\ln\left|\frac{50}{350}\right| \approx 0.08588.$$

It takes approximately 0.08588 years (or approximately 1.0306 months) for the population of creatures to reach 50.

c. [4 points] Determine if the integral $T(400) = \int_{10}^{400} \frac{20dx}{x(400-x)}$ converges or diverges. What does your conclusion mean in terms of the creatures on the preservation?

Solution:

$$\begin{aligned} T(400) &= \frac{1}{20}\int_{10}^{400} \frac{dx}{x} + \lim_{b\to 400} \frac{1}{20}\int_{10}^{b} \frac{dx}{400-x} \\ &= \frac{1}{20}\ln|40| + \lim_{b\to 400}\left(-\frac{1}{20}\ln|400-b| + \frac{1}{20}\ln|390|\right) \end{aligned}$$

We know that $\lim_{b\to 400}\left(-\frac{1}{20}\ln|400-b|\right)$ diverges, so the integral diverges. This means that the time to reach 400 creatures is infinite, so the population will never reach 400 creatures.