5. [12 points] A right isosceles triangle is a right triangle whose sides containing the right angle are of equal length. The length from the triangle's hypotenuse to its right-angle vertex (opposite of the hypotenuse) is half the length of the hypotenuse.
Consider the solid whose cross sections perpendicular to the x-axis are right isosceles triangles, where the hypotenuse of each triangular cross-section is contained in the region of the xy-plane bounded by the curves $y = \sin(x)$ and $y = -\sin(x)$ between $x = 0$ and $x = \pi$.

a. [3 points] Find the volume of the cross-sectional slice located at $x = x_i$ with thickness Δx.

Solution: The triangle has base length $2\sin(x_i)$ and the height length is $\sin(x_i)$. The area of the triangle is then $\sin^2(x_i)$, so the volume of the slice is $\sin^2(x_i)\Delta x$.

b. [3 points] Write a Riemann sum that approximates the volume of the entire solid using n cross-sectional slices.

Solution: We can approximate the area of the solid by finding the volume of n cross-sectional slices of depth Δx, and then adding these slices.

$$\text{volume} \approx \sum_{i=1}^{n} \sin^2(x_i)\Delta x$$

c. [6 points] Find the exact volume of the solid by using a definite integral.

Solution: As we let $n \to \infty$ in our Riemann sum, we approach the exact volume with a definite integral. We have volume $= \int_0^\pi \sin^2(x)dx$. We use integration by parts and some algebraic manipulation to solve the integral. Let $u = \sin(x), du = \cos(x)dx, dv = \sin(x)dx, v = -\cos(x)$. Then we have

$$\begin{aligned}
\int_0^\pi \sin^2(x)dx &= -\sin(x)\cos(x) + \int_0^\pi \cos^2(x)dx \\
&= -\sin(x)\cos(x)|_0^\pi + \int_0^\pi (1 - \sin^2(x))dx \\
&= 0 + \int_0^\pi dx - \int_0^\pi \sin^2(x)dx \\
2\int_0^\pi \sin^2(x)dx &= \int_0^\pi dx \\
2\int_0^\pi \sin^2(x)dx &= x|_0^\pi \\
\int_0^\pi \sin^2(x)dx &= \frac{\pi}{2}
\end{aligned}$$