8. [13 points] Let $C(u)$ be a function that satisfies $C'(u) = \frac{\cos(u^2)}{u}$, $C(2) = 3$, and let $S(u)$ be a function that satisfies $S'(u) = \frac{\sin(u^2)}{u}$, $S(2) = -1$.

a. [4 points] Write expressions for $C(t)$ and $S(t)$ that satisfy the above conditions.

Solution:

$$C(t) = \int_2^t \frac{\cos(u^2)}{u} du + 3 \qquad S(t) = \int_2^t \frac{\sin(u^2)}{u} du - 1$$

b. [5 points] A particle traces out the curve given by the parametric equations $x(t) = C(\ln(t))$, $y(t) = S(\ln(t))$ for $t \geq 10$. What is the speed of the particle at time t? You may assume that $t \geq 10$.

Solution:

$$\begin{aligned}
\text{speed} &= \sqrt{\left(\frac{d}{dt}\int_2^{\ln(t)} \frac{\cos(u^2)}{u} du\right)^2 + \left(\frac{d}{dt}\int_2^{\ln(t)} \frac{\sin(u^2)}{u} du\right)^2} \\
&= \sqrt{\left(\frac{1}{t} \cdot \frac{\cos(\ln(t))^2}{\ln(t)}\right)^2 + \left(\frac{1}{t} \cdot \frac{\sin(\ln(t))^2}{\ln(t)}\right)^2} \\
&= \frac{1}{t\ln(t)}
\end{aligned}$$

c. [4 points] For $t \geq 10$, is the curve given by the parametric equations in part (b) of finite or infinite length? Justify your answer.

Solution: The curve has infinite length. The arc length for $t \geq 10$ is given by integrating the speed over the interval $[10, \infty)$. With a u-substitution $u = \ln(t)$ we have

$$\text{arc length} = \int_{10}^{\infty} \frac{dt}{t\ln(t)} = \int_{\ln(10)}^{\infty} \frac{du}{u}$$

which diverges by the p-test.