8. [15 points] A metal plate with constant density $5\frac{kg}{m^2}$ has a shape bounded by the curve $y = 1 - x^2$ and the x-axis. You should evaluate all definite integrals in this problem by hand.

a. [3 points] Write an expression that approximates the amount of mass contained in a horizontal slice of the metal plate of width Δy meters, located at y_i meters from the x axis. Your answer should be be in terms of y_i and Δy.

Solution:

$m_i =$(density) (area of slice)$= (5)(2\sqrt{1-y_i}\Delta y)$.

b. [4 points] Find the exact total mass of the metal plate.

Solution:

$$m = \int_0^1 10\sqrt{1-y}dy = -\frac{20}{3}(1-y)^{\frac{3}{2}}\,|_0^1 = \frac{20}{3} = 6.66 \text{ kg}.$$

c. [8 points] Find the center of mass of the metal plate.

Solution:

$\bar{x}$=**0** by symmetry.

$$\bar{y} = \frac{\int_0^1 10y\sqrt{1-y}dy}{\frac{20}{3}} = \frac{3}{2}\int_0^1 y\sqrt{1-y}dy = \frac{3}{2}\left(\frac{4}{15}\right) = \frac{2}{5} = .4$$

With u substitution:

$$\int_0^1 y\sqrt{1-y}dy =$$

$$u = 1-y \qquad = -\int_1^0 (1-u)u^{\frac{1}{2}}du$$

$$= \frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\,|_0^1 = \frac{4}{15}$$

With integration by parts:

$$\int_0^1 y\sqrt{1-y}dy = -\frac{2}{3}y\sqrt{1-y}|_0^1 + \frac{2}{3}\int_0^1 (1-y)^{\frac{3}{2}}dy$$
$$= -\frac{2}{3}\left(\frac{2}{5}(1-y)^{\frac{5}{2}}\right)|_0^1 = \frac{4}{15}$$