5. [11 points] During a friendly game of ten-pin bowling, your friends Walter and Smokey begin to quarrel over whether Smokey’s toe slipped over the foul line. Meanwhile, you decide to pass the time by finding a mathematical model for the shape of a bowling pin. After some careful thought, you find that a fallen pin is a solid of revolution given by rotating the region under the curve

\[ B(x) = \sqrt{1.2 + 5.32x - 1.485x^2 + .135x^3 - .004x^4} \]

over the interval \([0, 15]\) about the \(x\)-axis. The region is pictured below. All measurements are in inches. A helpful stranger in the bowling alley informs you that the wood used to make the pin has density \(\delta = 17\) grams per cubic inch.

![Diagram of a bowling pin](image)

a. [3 points] Write a definite integral that gives the mass of the bowling pin. You do not need to evaluate this integral.

**Solution:** Since the bowling pin is a solid of revolution, the volume of a cylindrical slice located \(x\) inches from the base of the pin can be approximated by \(\pi B(x)^2 \Delta x\). Thus, the mass of the slice is approximately

\[
\delta \pi B(x)^2 \Delta x = 17\pi \left( \sqrt{1.2 + 5.32x - 1.485x^2 + .135x^3 - .004x^4} \right)^2 \Delta x,
\]

so that the mass of the entire pin is

\[
\int_0^{15} 17\pi B(x)^2 dx = \int_0^{15} 17\pi \left( 1.2 + 5.32x - 1.485x^2 + .135x^3 - .004x^4 \right) dx \text{ grams}.
\]

b. [6 points] What are the coordinates \((\bar{x}, \bar{y})\) of the bowling pin’s center of mass? You may use your calculator to answer this question.

**Solution:** Since the bowling pin has uniform density, we know immediately from rotational symmetry that \(\bar{y} = 0\). Using the formula for the \(x\)-coordinate of the center of mass, we have

\[
\bar{x} = \frac{\int_0^{15} \delta x\pi B(x)^2 dx}{\int_0^{15} \delta \pi B(x)^2 dx} = \frac{\int_0^{15} 17x\pi \left( 1.2 + 5.32x - 1.485x^2 + .135x^3 - .004x^4 \right) dx}{\int_0^{15} 17\pi \left( 1.2 + 5.32x - 1.485x^2 + .135x^3 - .004x^4 \right) dx} = 5.
\]

Thus, the coordinates of the center of mass of the bowling pin are \((5, 0)\), where each coordinate is measured in inches.
c. [2 points] Suppose the wood used to make the pin had density $\delta = 16$ grams per cubic inch. How does this affect the position $(\bar{x}, \bar{y})$ of the center of mass?

Solution: The center of mass is not affected since the integral

$$\bar{x} = \frac{\int_0^{15} \delta x \pi B(x)^2 dx}{\int_0^{15} \delta \pi B(x)^2 dx} = \frac{\int_0^{15} x \pi B(x)^2 dx}{\int_0^{15} \pi B(x)^2 dx}$$

is independent of $\delta$. The same is true for $\bar{y}$. 