7. [13 points] Household electricity in the United States is supplied in the form of an alternating current that varies sinusoidally with a frequency of 60 cycles per second (Hz). The voltage is given by the equation

$$E(t) = 170\sin(120\pi t),$$

where t is given in seconds and E is in volts.

a. [7 points] Using integration by parts, find $\int \sin^2\theta d\theta$. Show all work to receive full credit. (Hint: $\sin^2\theta + \cos^2\theta = 1$.)

Solution: We first note that $\int \sin^2\theta d\theta = \int \sin\theta(\sin\theta)d\theta$, so that we may take $u = \sin\theta, dv = \sin\theta d\theta$ (and $du = \cos\theta d\theta, v = -\cos\theta$). Then integration by parts gives

$$\begin{aligned}\int \sin^2\theta d\theta &= \sin\theta(-\cos\theta) - \int -\cos\theta(\cos\theta)d\theta \\ &= -\sin\theta\cos\theta + \int \cos^2\theta d\theta.\end{aligned}$$

Using the trig. identity given in the hint, we obtain

$$\int \sin^2\theta d\theta = -\sin\theta\cos\theta + \int \left(1-\sin^2\theta\right) d\theta = -\sin\theta\cos\theta + \int d\theta - \int \sin^2\theta.$$

The integral on the far right also appears on the left, so combining like terms, we get

$$\begin{aligned}2\int \sin^2\theta d\theta &= -\sin\theta\cos\theta + \int d\theta \\ \int \sin^2\theta d\theta &= \frac{1}{2}\left(-\sin\theta\cos\theta + \int d\theta\right) \\ &= -\frac{1}{2}\sin\theta\cos\theta + \frac{\theta}{2} + C.\end{aligned}$$

b. [6 points] Voltmeters read the root-mean-square (RMS) voltage, which is defined to be the square root of the average value of $[E(t)]^2$ over one cycle. Find the exact RMS voltage of household current.

Solution: Since the frequency of the current is 60 cycles per second, one cycle is completed every $\frac{1}{60}$ seconds. Thus

$$\text{RMS voltage} = \sqrt{\frac{1}{\frac{1}{60}-0}\int_0^{\frac{1}{60}} E(t)^2 dt}$$
$$= \sqrt{60\int_0^{\frac{1}{60}} 170^2 \sin^2(120\pi t)\, dt}.$$

Substituting $w = 120\pi t, dw = 120\pi dt$, we get

$$\text{RMS voltage} = \sqrt{60\int_{w(0)}^{w(\frac{1}{60})} 170^2 \sin^2(w) \cdot \frac{1}{120\pi} dw}$$
$$= \sqrt{\frac{170^2}{2\pi}\int_0^{2\pi} \sin^2(w)\, dw}.$$

Using the antiderivative we found in part (a) with $C = 0$, the Fundamental Theorem of Calculus gives

$$\text{RMS voltage} = \sqrt{\frac{170^2}{2\pi}\left(-\frac{1}{2}\sin\theta\cos\theta + \frac{\theta}{2}\right)\Big|_0^{2\pi}}$$
$$= \sqrt{\frac{170^2}{2\pi}\left((-\frac{1}{2}\sin(2\pi)\cos(2\pi) + \frac{2\pi}{2}) - (-\frac{1}{2}\sin(0)\cos(0) + \frac{0}{2})\right)} = \frac{170}{\sqrt{2}} \text{ Volts}.$$

Note that due to the periodicity of the sine function, the average value over one cycle could also have been computed over $0 \le t \le 1$ (or any other number of periods).