7. [6 points] Suppose that g is a continuous function, and define another function G by

$$G(x) = \int_0^x g(t)\,dt.$$

Given that $\int_0^7 g(x)\,dx = 5$, compute

$$\int_0^7 g(x)(G(x))^2\,dx.$$

Show each step of your computation.

Solution: Substitution gives

$$\int_0^7 g(x)(G(x))^2\,dx = \int_{G(0)}^{G(7)} u^2\,du = \left.\frac{u^3}{3}\right|_0^5 = \frac{125}{3}.$$

Alternatively, integrate by parts to obtain

$$\int_0^7 g(x)(G(x))^2\,dx = \left.(G(x))^3\right|_0^7 - 2\int_0^7 g(x)(G(x))^2\,dx,$$

which after rearranging gives

$$\int_0^7 g(x)(G(x))^2\,dx = \frac{1}{3}\left(\left.(G(x))^3\right|_0^7\right) = \frac{125}{3}.$$