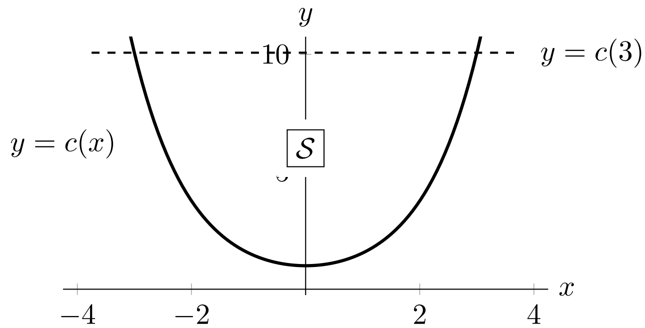
2. [13 points]

Consider the function c defined for all real numbers x by the formula

$$c(x) = \frac{e^x + e^{-x}}{2}.$$

A portion of the graph of this "catenary" function is shown as the solid curve in the graph on the right.



Let $\mathcal{S}$ be the region bounded by the graph of $y = c(x)$ and the line $y = c(3)$. This region $\mathcal{S}$ is shown in the figure above.

a. [2 points] Write, but do **not** evaluate, an expression involving one or more integrals that gives the area of $\mathcal{S}$.

Solution: Note that $c(3) = \frac{e^3+e^{-3}}{2}$ is a constant and that the graph of $y = c(3)$ intersects the graph of $y = c(x)$ at $x = \pm 3$. The area of $\mathcal{S}$ is the area between the graphs above.

$$\text{Area} = \int_{-3}^{3} \frac{e^3 + e^{-3}}{2}\,dx - \int_{-3}^{3} \frac{e^x + e^{-x}}{2}\,dx = \int_{-3}^{3} \left(\frac{e^3 + e^{-3}}{2} - \frac{e^x + e^{-x}}{2} \right) dx$$

b. [5 points] A solid is obtained by rotating the region $\mathcal{S}$ about the x-axis. Write, but do **not** evaluate, an expression involving one or more integrals that gives the volume of this solid.

Solution: Taking slices perpendicular to the x-axis, each slice is "washer"-shaped, and the volume of such a slice at x of thickness Δx is approximately $\pi \left((c(3))^2 - (c(x))^2 \right) \Delta x$. The volume of the entire solid is then

$$\int_{-3}^{3} \pi \left((c(3))^2 - (c(x))^2 \right) dx = \int_{-3}^{3} \pi \left[\left(\frac{e^3 + e^{-3}}{2} \right)^2 - \left(\frac{e^x + e^{-x}}{2} \right)^2 \right] dx$$

c. [3 points] Write, but do **not** evaluate, an expression involving one or more integrals that gives the arc length of the graph of $y = c(x)$ over the interval $-3 \le x \le 3$. (Your answer should **not** involve any function names.)

Solution: We first compute c' to find $c'(x) = \dfrac{e^x - e^{-x}}{2}$.

Substituting this into the formula for the arc length of the graph of a function gives

$$\text{Arc length} = \int_{-3}^{3} \sqrt{1 + (c'(x))^2}\,dx = \int_{-3}^{3} \sqrt{1 + \left(\frac{e^x - e^{-x}}{2} \right)^2}\,dx.$$

d. [3 points] Will the midpoint rule with 2000 subdivisions give an underestimate or an overestimate of the value of $\displaystyle\int_{-3}^{0} c(x)\,dx$?

Circle your answer below. Then briefly explain your reasoning in the space on the right.

Circle one:

$\boxed{\textbf{Underestimate}}$

Overestimate

Neither (They are equal)

Cannot be determined

Briefly explain your reasoning.

Solution: Note that the second derivative of c is given by $c''(x) = \frac{e^x+e^{-x}}{2}$, which is positive for all values of x (since both e^x and e^{-x} are always positive). Since c'' is always positive, the function c is always concave up. Therefore every approximation of this definite integral using the midpoint rule (including MID(2000)) will be an underestimate.†

†Note that $c'' = c$. This is a very special property of this function c.