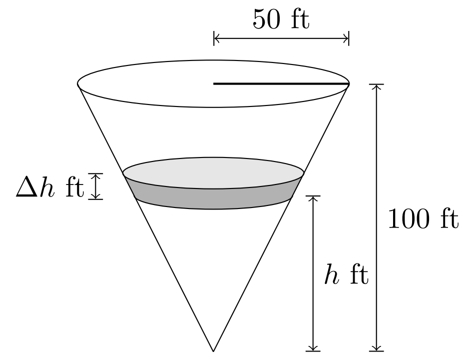
4. [10 points]

A farming cooperative stores its alfalfa seed in a giant funnel. The funnel is in the shape of a right circular cone with height 100 feet and radius 50 feet at the top. A diagram of such a cone is shown in the figure on the right.



a. [4 points] Write an expression in terms of h that approximates the volume (in cubic feet) of a horizontal slice of the funnel of thickness Δh feet at a height of h feet above the bottom of the funnel. (Assume Δh is positive but very small.)

Solution: Using similar triangles, a horizontal cross section at height h will be a circle having radius r given by the proportion $\frac{50}{100} = \frac{r}{h}$. Hence $r = h/2$.
The approximate volume of such a slice is then

$$V_{slice} \approx \pi r^2 \Delta h = \pi (h/2)^2 \Delta h \quad \text{ft}^3$$

b. [6 points] For parts i and ii below, assume that the funnel is full of alfalfa seed. The funnel is clogged, so the alfalfa seed must be removed from above in order to clear the clog. Assume that alfalfa seed weighs 48 pounds per cubic foot.

i. Using your answer to part (a), write an expression in terms of h that approximates the work, in foot-pounds, done in moving a horizontal slice of seed of thickness Δh that is h feet above the bottom of the funnel to the top of the funnel.

Solution: The weight of such a slice is

$$\text{Weight}_{slice} = 48 \cdot (V_{slice}) \approx 48(\pi (h/2)^2 \Delta h) \quad \text{lbs}$$

This slice needs to be moved up a distance of about $100 - h$ feet. Hence the work done on such a slice is

$$\text{Work}_{slice} \approx 48(100 - h)(\pi (h/2)^2 \Delta h) \quad \text{ft-lbs}$$

ii. Write, but do **not** evaluate, an expression involving one or more integrals that gives the total work, in foot-pounds, that must be done to empty the tank of seed.

Solution:

$$\text{Work}_{\text{total}} = \int_0^{100} 48(100 - h)(\pi (h/2)^2)\, dh = \int_0^{100} 12\pi h^2 (100 - h)\, dh \quad \text{ft-lbs}$$