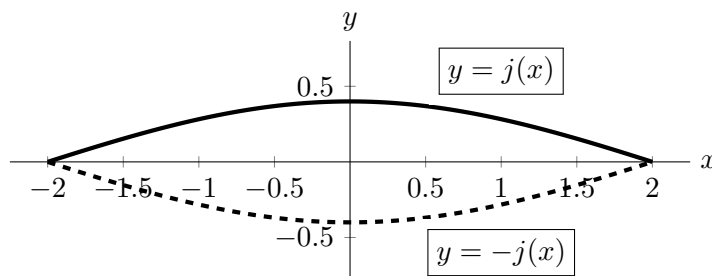


6. [8 points] A rattleback top is a toy that exhibits interesting physical properties. The toy can be modeled by a solid whose base is the region between the graphs of $j(x)$ and $-j(x)$, shown below. The cross sections perpendicular to the x -axis are semicircles.



The graph of $j(x)$ is solid, the graph of $-j(x)$ is dashed, and the units on both axes are centimeters. Both graphs are bounded between the vertical lines $x = -2$ and $x = 2$.

- a. [5 points] Set up, but do **not** evaluate, an expression involving one or more integrals that gives the volume, in cubic centimeters, of the solid rattleback top. Your answer may involve the function name j .

Solution: By slicing perpendicular to the x -axis, a cross-section at horizontal coordinate x will have height $j(x) - (-j(x)) = 2j(x)$. This is the diameter of our semicircular cross-section, so the radius of the cross section is then half the diameter, or $\frac{2j(x)}{2} = j(x)$. A semicircle of radius r has area $\frac{\pi}{2}r^2$, so if each slice has thickness Δx , then the volume of such a slice can be approximated by

$$V_{\text{slice}} \approx \frac{\pi}{2}(j(x))^2 \Delta x.$$

We compute the total volume by summing over all slices and taking the limit as the thickness of the slices approaches 0. The total volume is then

$$V_{\text{total}} = \int_{-2}^2 \frac{\pi}{2}(j(x))^2 dx \text{ cm}^3.$$

- b. [3 points] In order to make the rattleback top spin like a top, it is made out of plastic that has a mass density given by the function $\delta(x)$ grams per cubic centimeter, where x is the x -coordinate in the diagram above. Set up, but do **not** evaluate, an expression involving one or more integrals that gives the mass, in grams, of the rattleback top. Your answer may involve the function names j and/or δ .

Solution: For small Δx , the mass of a slice as described in part a is approximately

$$m_{\text{slice}} \approx \frac{\pi}{2}(j(x))^2 \delta(x) \Delta x.$$

Hence the total mass is

$$m_{\text{total}} = \int_{-2}^2 \frac{\pi}{2}(j(x))^2 \delta(x) dx \text{ grams}.$$