10. [8 points] Two functions, $f(x)$ and $g(x)$ are continuous and differentiable for all $x > 2$, and:

- $\lim_{x\to 2+} f(x) = \infty$ (this means that $f(x)$ has a vertical asymptote at $x = 2$),
- $\frac{d}{dx}\left(\frac{3 - 3\cos(\pi x)}{g(x)}\right) = f(x)$ for all $x > 2$,
- $g(3) = 4$,
- $\lim_{x\to 2+} g(x) = 0$, and
- $\lim_{x\to 2+} g'(x) = 10$.

Determine whether the following integral converges or diverges, and if the integral converges, give its exact value. Be sure to show all work and indicate any theorems you use.

$$\int_2^3 f(x)dx$$

Answer *(Circle one):* **Diverges** **Converges** to: **3/2**

Justification:

Solution:
The first bullet point tells us that this integral is improper, so we must start by changing to limit notation.

$$\begin{aligned}\int_2^3 f(x)dx &= \lim_{b\to 2^+}\int_b^3 f(x)dx\\ &= \lim_{b\to 2^+}\left(\frac{3-3\cos(\pi 3)}{g(3)} - \frac{3-3\cos(\pi b)}{g(b)}\right)\\ &= \lim_{b\to 2^+}\left(\frac{6}{4} - \frac{3-3\cos(\pi b)}{g(b)}\right)\\ &= \frac{3}{2} - \lim_{b\to 2^+}\frac{3-3\cos(\pi b)}{g(b)}\end{aligned}$$

At this point, we see that $\lim_{b\to 2^+} g(b) = 0$ and $\lim_{b\to 2^+} 3-3\cos(\pi b) = 0$, so we will apply L'Hopital's Rule to get

$$\begin{aligned}\frac{3}{2} - \lim_{b\to 2^+}\frac{3-3\cos(\pi b)}{g(b)} &= \frac{3}{2} - \lim_{b\to 2^+}\frac{3\sin(\pi b)}{g'(b)} \text{ by L'Hopital's Rule}\\ &= \frac{3}{2} - \frac{0}{10}\\ &= \frac{3}{2}\end{aligned}$$