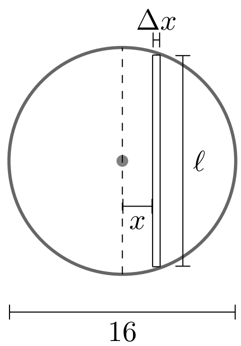
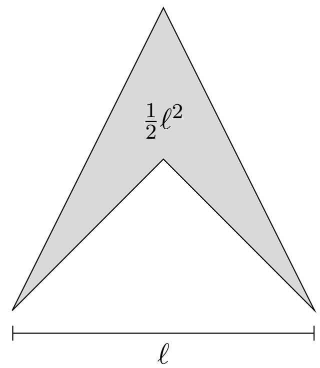
5. [11 points] Tony's climbing gym wants to put in a climbing structure based off of the Chicago Bean. However, they want to make it more angular. The base of the structure will be in the shape of a circle with an 8 meter radius. The cross-sections perpendicular to the circle lying above a slice of the circle of length ℓ meters (as shown below) have area $\frac{1}{2}\ell^2$ square meters and are pictured below. The density of the material used to build the structure is not constant and has density dependent on its horizontal distance x from the vertical diameter through the circle. The density in kg/m^3 is given by $\delta(x) = 1000\sqrt{1+x^2}$.





a. [2 points] Write an expression that gives the quantity ℓ in terms of x.

Solution: Using x and ℓ as defined in the figure above, we have $x^2 + \left(\frac{\ell}{2}\right) = 8^2$. Solving this, since $\ell \geq 0$,

$$\ell = 2\sqrt{64 - x^2}.$$

b. [3 points] Write an expression that gives the approximate volume, in cubic meters, of a slice of the structure a horizontal distance x meters away from the diameter of the circle with thickness Δx. Your expression should not involve any integrals.

Solution: The approximate volume of a slice of the structure x meters away from the vertical diameter of the circle, using (a) is

$$\frac{1}{2}\ell^2\Delta x = 2(64 - x^2)\Delta x.$$

c. [3 points] Using your expression from (b) to write an expression involving integrals which gives the total volume of the structure in cubic meters. Do not evaluate any integrals.

Solution: The volume is the integral of the volume of the slice in (b) as $\Delta x \to 0$, from $x = -8$ to $x = 8$ (or alternatively twice the integral from $x = 0$ to $x = 8$):

$$2\int_0^8 2(64 - x^2)dx = 4\int_0^8 64 - x^2 dx.$$

d. [3 points] Write an expression involving integrals which gives the total mass of the structure in kg. Your answer may contain $\delta(x)$. Do not evaluate any integrals.

Solution: Using our answer from (b) and the density of the slice a horizontal distance x meters away from the vertical diameter of the circle, the approximate mass of a slice of the structure is

$$2\delta(x)(64 - x^2)\Delta x.$$

The mass is the integral of the mass of the slice as $\Delta x \to 0$ from $x = -8$ to $x = 8$ (or alternatively twice the integral from $x = 0$ to $x = 8$ since $\delta(x)$ is even):

$$4\int_0^8 \delta(x)(64 - x^2)dx.$$