6. [10 points] Denise and Trystan are underwater research scientists, and they are preparing to
descend into the ocean in a newly-constructed submarine. The submarine’s shape is given by
rotating the region below the curve \( y = \sqrt{x + 1} \), above the \( x \)-axis, and between \( x = 0 \) and
\( x = 10 \) (see figure) about the \( x \)-axis. Here, \( x \) and \( y \) are measured in meters.

Graph of \( y = \sqrt{x + 1} \) from \( x = 0 \) to \( x = 10 \)

The density of the submarine is not constant, due to the advanced materials used in its
construction. Instead, the density \( p(x) \) varies, and is given by \( p(x) = (x - 5)^2 + 1 \text{ kg/m}^3 \).

a. [5 points] Write an expression for the volume of a slice of the submarine at position \( x \)
and of thickness \( \Delta x \). Include units.

Solution: The radius of such a slice is given by \( r(x) = \sqrt{x + 1} \), so the volume is
\[ \pi (r(x))^2 \Delta x = \pi (x + 1) \Delta x \text{ m}^3. \]

b. [2 points] Write an expression for the mass of the slice you found in part (a). Include
units.

Solution: The density function \( p(x) \) depends only on \( x \), so the density is roughly constant
on the slice from part (a), as long as \( \Delta x \) is very small. The mass of such a slice is then
\[ M(x) = p(x) \cdot \pi (x + 1) \Delta x = [(x - 5)^2 + 1] \pi (x + 1) \Delta x \text{ kg}. \]

c. [3 points] Write, but do not evaluate, an integral which gives the total mass of the
submarine. Include units.
Solution: The approximate mass of the submarine is obtained by adding together all the masses of the slices calculated above to get $\sum[(x - 5)^2 + 1]\pi (x + 1)\Delta x$. In the limit we get the exact mass in the form of an integral:

$$\int_{0}^{10} [(x - 5)^2 + 1]\pi (x + 1) \, dx.$$