8. [12 points]

a. [6 points] Split the function

$$f(x) = \frac{x+2}{(x-2)^2(x-1)}$$

into partial fractions. Do not integrate your result. Please show all of your work.

Solution: Start by splitting:

$$\frac{x+2}{(x-2)^2(x-1)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}.$$

By multiplying through to get a common denominator, we get

$$x+2 = A(x-2)^2 + B(x-1)(x-2) + C(x-1). \tag{1}$$

Method 1 (Comparing coefficients): we multiply out the products on the right hand side and group terms which have the same power of x in them. This gives:

$$x+2 = (A+B)x^2 + (-4A-3B+C)x + (4A+2B-C). \tag{2}$$

This gives us the system of equations:

$$A+B=0, \quad -4A-3B+C=1, \quad 4A+2B-C=2.$$

We solve this system to obtain values: $A=3, B=-3, C=4$.

Method 2 (Plugging in values): If we plug $x=2$ into (1) we get

$$2+2 = A(2-2)^2 + B(2-1)(2-2) + C(2-1),$$

which simplifies to $4=C$.
If we plug $x=1$ into (1) we get

$$3 = A(1-2)^2 + B(1-1)(1-2) + C(1-1)$$

which simplifies to $3=A$.
If we plug these values for A and C back into (1) and also plug in $x=3$ we obtain the equation

$$\begin{aligned} 3+2 &= 3(3-2)^2 + B(3-1)(3-2) + 4(3-1) \\ 5 &= 3+2B+8 \\ -6 &= 2B \\ B &= -3. \end{aligned}$$

So we find $A=3, B=-3, C=4$.

8. (continued)

b. [6 points] Given the partial fraction decomposition

$$\frac{-3x}{(x+1)(x^2+1)} = \frac{3}{2(x+1)} - \frac{3(x+1)}{2(x^2+1)},$$

evaluate the following indefinite integral, showing all of your work:

$$\int \frac{-3x}{(x+1)(x^2+1)}\,dx.$$

Solution: Start by splitting up the integral:

$$\int \frac{-3x}{(x+1)(x^2+1)}\,dx = \int \frac{3}{2(x+1)}\,dx - \int \frac{3(x+1)}{2(x^2+1)}\,dx.$$

Then we split up the second integral to get

$$\int \frac{-3x}{(x+1)(x^2+1)}\,dx = \int \frac{3}{2(x+1)}\,dx - \int \frac{3x}{2(x^2+1)}\,dx - \int \frac{3}{2(x^2+1)}\,dx.$$

For the first integral, we have:

$$\int \frac{3}{2(x+1)}\,dx = \frac{3}{2}\int \frac{1}{x+1}\,dx = \frac{3}{2}\ln|x+1| + C.$$

For the second integral, we u-substitution with $u = x^2+1$, and $du = 2xdx$, so:

$$\int \frac{3x}{2(x^2+1)}\,dx = \frac{3}{2}\int \frac{x}{x^2+1}\,dx = \frac{3}{4}\int \frac{1}{u}\,du = \frac{3}{4}\ln|x^2+1| + C.$$

For the final integral, we have:

$$\int \frac{3}{2(x^2+1)}\,dx = \frac{3}{2}\int \frac{1}{x^2+1}\,dx = \frac{3}{2}\arctan(x) + C.$$

Putting this all together, we get

$$\int \frac{-3x}{(x+1)(x^2+1)}\,dx = \frac{3}{2}\ln|x+1| - \frac{3}{4}\ln|x^2+1| - \frac{3}{2}\arctan(x) + C.$$