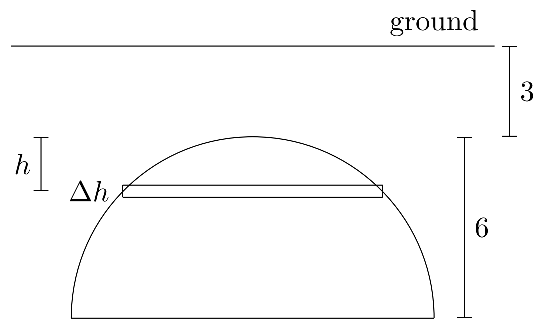
8. [15 points] A town's local cheese dispensary has a cheese tank that is located 3 meters below ground level. The cheese tank is in the shape of a **hemisphere with radius 6 meters**. The diagram below shows a cross-section of the tank below the ground.
Assume that the density of the cheese in the tank is given by the function $\delta(h)$ (measured in kilograms per cubic meter), where h is measured in meters from the **top of the tank**. You may assume that the acceleration due to gravity is $g = 9.8\text{m/s}^2$.



a. [5 points] Consider a horizontal slice of cheese, h meters from the top of the tank with a small thickness of Δh meters, as depicted in the diagram above. Write an expression which approximates the mass of this slice as a function of h. Your answer may include $\delta(h)$. Your answer should not involve any integrals. Include units.

Solution: Horizontal slices as shown in the figure above are approximately cylindrical. The radius of each slice, r, relates to the radius of tank and h via the right-triangle whose legs are $6 - h$ and r, and whose hypotenuse is 6. Thus, the radius of the slice is located h units from the top of the tank is $\sqrt{6^2 - (6-h)^2}$. The mass of the slice is given approximately by $\pi\, r^2\, \delta(h)\, \Delta h$, measured in kilograms.

Answer: $\pi\,(6^2 - (6-h)^2)\,\delta(h)\,\Delta h$ **Units:** kg

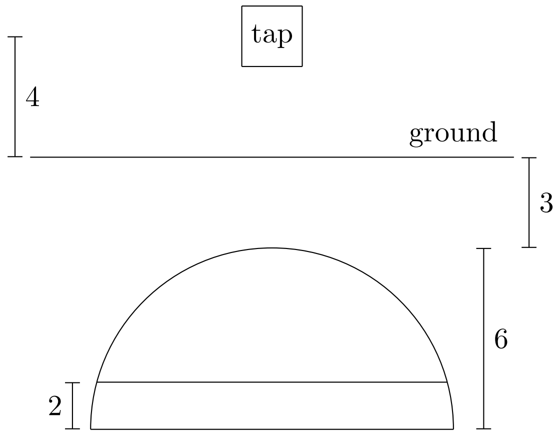
b. [5 points] Assume that the tank is entirely filled with cheese. Write an expression involving one or more integrals that gives the work done to pump all the cheese in the tank up to ground level. Your answer may include $\delta(h)$. Do not evaluate your integral(s). Include units.

Solution: To find the weight of the slice in the picture above, we should take the mass we found in part (a), and multiply by g. This slice is moved a distance of h meters to the top of the tank, and then a further 3 meters, for a total distance of $3 + h$ meters. The work done on this slice is then $\pi\,(9.8)(6^2 - (6-h)^2)\,\delta(h)(3+h)\,\Delta h$, and so integrating from $h = 0$ to $h = 6$ gives our answer. The units for work in this context are joules.

Answer: $\displaystyle\int_0^6 \pi\,(9.8)\,(6^2 - (6-h)^2)\,\delta(h)\,(3+h)\,\mathrm{d}h$ **Units:** J

8. (continued)

c. [5 points] Now assume that the tank is only filled up to a depth of 2 meters with cheese. The dispensary has a tap to the tank that is located 4 meters above ground level. The diagram below depicts the tank of cheese and the position of the tap. Write an expression involving one or more integrals that gives the work done to pump all the cheese in the tank up to the tap. Your answer may involve $\delta(h)$. Do not evaluate your integral(s). Include units.



Solution: To modify our answer from part (b), first notice that the slice must travel an extra 4 meters, and so we replace the old distance of $3 + h$ with $7 + h$. Then note that since the tank is only partially filled, the values of h for a slice range between 4 and 6. This gives us the new bounds on our integral. The units are the same as before.

Answer: $\int_4^6 \pi\,(9.8)\,(6^2 - (6-h)^2)\,\delta(h)\,(7+h)\,\mathrm{d}h$ **Units:** J