4. [8 points] Consider the following function:

$$F(x) = \int_1^{\ln x} \frac{\cos^2(t)}{t}\, dt.$$

a. [2 points] Find a value of a such that $F(a) = 0$. Show your work.

Solution: $F(a) = 0$ when the upper and lower bounds are equal, in other words when $\ln(a) = 1$. This happens when $a = e$.

Answer: $a =$ e

b. [3 points] Calculate $F'(x)$.

Solution: We can use the chain rule to find $F'(x)$:

$$F'(x) = \frac{\cos^2(\ln(x))}{\ln(x)} \frac{d}{dx}(\ln x) = \frac{\cos^2(\ln(x))}{\ln(x)} \frac{1}{x} = \frac{\cos^2(\ln(x))}{x \ln(x)}.$$

Answer: $F'(x) =$ $\dfrac{\cos^2(\ln(x))}{x \ln(x)}$

c. [3 points] Find a function $f(t)$ and constants a and C so that we may rewrite $F(x)$ in the form $\displaystyle\int_a^x f(t)\, dt + C$. There may be more than one correct answer.

Solution: From our earlier work, we know that $F(x)$ is an antiderivative of $F'(x) = \dfrac{\cos^2(\ln(x))}{x \ln(x)}$ which satisfies $F(e) = 0$. Using the Second Fundamental Theorem of Calculus, we see that we may express:

$$F(x) = \int_e^x \frac{\cos^2(\ln(t))}{t \ln(t)}\, dt + 0$$

$f(t) =$ $\dfrac{\cos^2(\ln(t))}{t \ln(t)}$ $a =$ e $C =$ 0