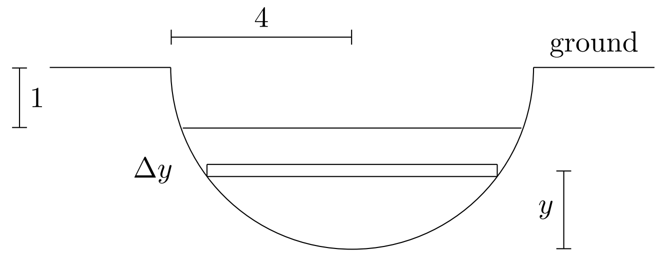
9. [8 points] Marcy's gigantic bird bath attracts lots of birds to her garden. The bath is carved out of the ground, in the shape of a **hemisphere** with radius 4 meters. A cross-section of the bath is depicted below. The bath is partially filled with muddy water, so that the surface of the water is 1 meter below ground level. The density of the water in the bath is given by the function $\delta(y)$ (measured in kilograms per cubic meter), where y is measured in meters from the **bottom of the bath**. You may assume that the acceleration due to gravity is $g = 9.8\text{m/s}^2$.



a. [4 points] Consider a horizontal slice of muddy water, y meters from the bottom of the bath with a small thickness of Δy meters, as depicted in the diagram above. Write an expression which approximates the mass, in kilograms, of this slice as a function of y. Your answer may include $\delta(y)$. Your answer should **not** involve any integrals.

Solution: Let r be the radius of the slice at height y. Using the Pythagorean Theorem, $r = \sqrt{16 - (4-y)^2}$. Therefore, the mass of the slice is

$$\delta(y)\pi r^2 \Delta y = \delta(y)\pi\left(16 - (4-y)^2\right)\Delta y = \delta(y)\pi\left(8y - y^2\right)\Delta y$$

Answer: $\delta(y)\pi\left(16 - (4-y)^2\right)\Delta y = \delta(y)\pi\left(8y - y^2\right)\Delta y$

b. [4 points] Write an expression involving one or more integrals that gives the work done, in joules, to pump all the water in the bath up to ground level. Do not evaluate your integral(s).

Solution: Using our expression for the mass of a slice from part a., we see that the weight of a slice, in newtons, is:

$$\delta(y)\pi g\left(8y - y^2\right)\Delta y$$

A horizontal slice at height y will need to be lifted a distance of $4 - y$ meters. Hence the work done to move such a slice is

$$\left(\delta(y)\pi g\left(8y - y^2\right) g\Delta y\right)(4-y) = \delta(y)\pi g\left(8y - y^2\right)(4-y)\Delta y$$

The water fills the bath up to height 3 meters, so the total work done to pump all the water in the bath up to ground level is

$$\int_0^3 \delta(y)\pi g\left(8y - y^2\right)(4-y)\,dy$$

Answer: $\int_0^3 \delta(y)\pi g\left(8y - y^2\right)(4-y)\,dy$