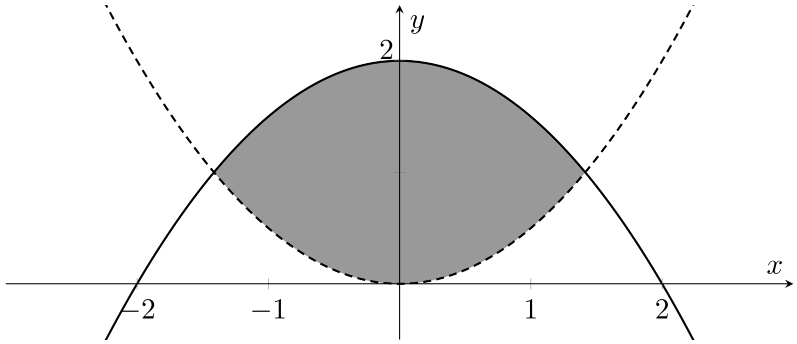
6. [9 points] A video game designer wants to model the shape of a mountain. The base of the mountain is the shaded region depicted below, bounded by the curves $y = 2 - \frac{x^2}{2}$ and $y = \frac{x^2}{2}$.



a. [5 points] Write an expression involving one or more integrals for the volume of a mountain whose base is the shaded region, and whose cross-sections perpendicular to the x-axis are semicircles. **Do not** evaluate any integrals in your expression.

Solution: First of all, x coordinates of points of intersection of the two curves are given by solutions of the equation $2 - \frac{x^2}{2} = \frac{x^2}{2}$ which are $x = -\sqrt{2}$, and $x = \sqrt{2}$.
Taking a slice x units from the y-axis, the *diameter* of the semicircle is given by

$$2 - \frac{x^2}{2} - \frac{x^2}{2} = 2 - x^2,$$

so the area of the semicircular prism with width Δx is

$$\frac{\pi}{2}\left(\frac{2-x^2}{2}\right)^2 \Delta x.$$

Integrating from $-\sqrt{2}$ to $\sqrt{2}$ should then give the total volume.

Answer: $\int_{-\sqrt{2}}^{\sqrt{2}} \frac{\pi}{2}\left(\frac{2-x^2}{2}\right)^2 \, dx$

b. [4 points] Determine the perimeter of the shaded region. Write an expression that involves one or more integrals. **Do not** evaluate any integrals in your expression.

Solution: Exploiting symmetry, we calculate the arclength of one of the curves and multiply by 2. Let us denote the top curve by

$$f(x) = 2 - \frac{x^2}{2}$$

The arclength of the top curve is then

$$\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{1 + [f'(x)]^2} \, dx = \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{1 + x^2} \, dx,$$

which we can double to find the total perimeter.

Answer: $2\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{1 + x^2} \, dx$