7. (10 pts) Using integration by parts, calculate

$$\int e^{-x}\cos(x)\,dx.$$

Let I be the integral in question. Then use integration by parts, with

$$\begin{aligned} u &= e^{-x} & v &= \sin(x) \\ u' &= -e^{-x} & v' &= \cos(x) \end{aligned}$$

to get

$$\begin{aligned} I &= \int uv'\,dx = uv - \int u'v\,dx = e^{-x}\sin(x) - \int -e^{-x}\sin(x)\,dx \\ &= e^{-x}\sin(x) + \int e^{-x}\sin(x)\,dx. \end{aligned}$$

Now do it again, with

$$\begin{aligned} u &= e^{-x} & v &= -\cos(x) \\ u' &= -e^{-x} & v' &= \sin(x) \end{aligned}$$

and get

$$\begin{aligned} I &= e^{-x}\sin(x) + \int uv'\,dx = e^{-x}\sin(x) + uv - \int u'v\,dx \\ &= e^{-x}\sin(x) + e^{-x}(-\cos(x)) - \int -e^{-x}(-\cos(x))\,dx \\ &= e^{-x}(\sin(x) - \cos(x)) - I. \end{aligned}$$

We got back where we started from, but with a minus sign, so we're OK. Now move the new I over to the other side and divide by 2 to get

$$I = \boxed{\frac{e^{-x}(\sin(x) - \cos(x))}{2} + C}.$$

Check by taking the derivative:

$$\begin{aligned} \frac{d}{dx}\left(\frac{e^{-x}(\sin(x) - \cos(x))}{2}\right) &= \frac{1}{2}\left(e^{-x}(\cos(x) - (-\sin(x))) + (-e^{-x})(\sin(x) - \cos(x))\right) \\ &= \frac{1}{2}e^{-x}(\cos(x) + \sin(x) - \sin(x) + \cos(x)) \\ &= e^{-x}\cos(x) \end{aligned}$$

as expected.