4. (15 pts.) For each of the following statements about a continuous function, f, circle **T** if the statement is always true, and otherwise circle **F**. If a statement is always true, explain why. If a statement is not always true, give an example of a function so that the statement is not true for that function.

(a) $\int xf(x)\,dx = x\int f(x)\,dx.$ **T** $\boxed{\textbf{F}}$

The function $f(x) = x$ is a counterexample to the statement. For, if $f(x) = x$, the left hand side is $\frac{x^3}{3} + C$ while the right hand side is $x(\frac{x^2}{2} + C_1) \neq \frac{x^3}{3} + C$.

(b) Every function, $f(x)$, that is continuous on an interval, $[a, b]$, has an antiderivative on that interval. $\boxed{\textbf{T}}$ **F**

This is a consequence of the *Construction Theorem for Antiderivatives*, Theorem 6.2 on page 279 of the text. By that theorem, an antiderivative for f is the function $F(x) = \int_a^x f(t)\,dt$.

(c) If f is a positive continuous function for $x \geq 0$ and if $\lim_{x\to\infty} f(x) = 0$, then $\int_1^\infty f(x)\,dx$ converges. **T** $\boxed{\textbf{F}}$

The function $f(x) = 1/x$ is a counterexample to the statement. If $f(x) = \frac{1}{x}$, then $\lim_{x\to\infty} f(x) = 0$ but $\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\ln(b)$ which is not a finite number. So the integral diverges.

5. (8 points). If F is the function defined for $x > 0$ by $F(x) = \int_1^x \frac{e^t}{t}\,dt$, show that $\int F(x)\,dx = xF(x) - e^x + C.$

We want to show the right hand side is an antiderivative for F; that is $\frac{d}{dx}(xF(x) - e^x + C) = F(x)$.

However, $\frac{d}{dx}(xF(x) - e^x + C) = F(x) + xF'(x) - e^x$ by the product rule.

$F'(x) = \frac{e^x}{x}$ by the F.T.C.

So $\frac{d}{dx}(xF(x) - e^x + C) = F(x) + x\frac{e^x}{x} - e^x = F(x)$ as required.