4. (15 pts.) For each of the following statements about a continuous function, \( f \), circle T if the statement is always true, and otherwise circle F. If a statement is always true, explain why. If a statement is not always true, give an example of a function so that the statement is not true for that function.

(a) \( \int xf(x) \, dx = x \int f(x) \, dx. \) \[ \begin{array}{c|c} & T & F \\ \hline \text{The function } f(x) = x \text{ is a counterexample to the statement. For, if } f(x) = x, & \end{array} \]

the left hand side is \( \frac{x^3}{3} + C \) while the right hand side is \( x\left(\frac{x^2}{2} + C_1\right) \neq \frac{x^3}{3} + C. \)

(b) Every function, \( f(x) \), that is continuous on an interval, \([a, b]\), has an antiderivative on that interval. \[ \begin{array}{c|c} & T & F \\ \hline \text{This is a consequence of the Construction Theorem for Antiderivatives, Theorem 6.2 on page 279 of the text. By that theorem, an antiderivative for } f & \end{array} \]

is the function \( F(x) = \int_a^x f(t) \, dt. \)

(c) If \( f \) is a positive continuous function for \( x \geq 0 \) and if \( \lim_{x \to \infty} f(x) = 0 \), then \( \int_1^\infty f(x) \, dx \) converges. \[ \begin{array}{c|c} & T & F \\ \hline \text{The function } f(x) = \frac{1}{x} \text{ is a counterexample to the statement. If } f(x) = \frac{1}{x}, & \end{array} \]

then \( \lim_{x \to \infty} f(x) = 0 \) but \( \int_1^\infty \frac{1}{x} \, dx = \lim_{b \to \infty} \int_1^b \frac{1}{x} \, dx = \lim_{b \to \infty} \ln(b) \) which is not a finite number. So the integral diverges.

5. (8 points). If \( F \) is the function defined for \( x > 0 \) by \( F(x) = \int_1^x \frac{e^t}{t} \, dt \), show that

\[ \int F(x) \, dx = xF(x) - e^x + C. \]

We want to show the right hand side is an antiderivative for \( F \); that is

\( \frac{d}{dx}(xF(x) - e^x + C) = F(x). \)

However, \( \frac{d}{dx}(xF(x) - e^x + C) = F(x) + xF'(x) - e^x \) by the product rule.

\( F'(x) = \frac{e^x}{x} \) by the F.T.C.

So \( \frac{d}{dx}(xF(x) - e^x + C) = F(x) + x\frac{e^x}{x} - e^x = F(x) \) as required.