6. (10 points) (a) Explain why \( \int_{0}^{\pi/3} \frac{\cos x}{\sin^2 x} \, dx \) is an improper integral.

Because the integrand, \( \frac{\cos(x)}{\sin^2(x)} \), is unbounded for \( x \) near 0.

(b) Show how the improper integral in part (a) is defined mathematically.

\[
\int_{0}^{\pi/3} \frac{\cos x}{\sin^2 x} \, dx = \lim_{a \to 0^+} \int_{a}^{\pi/3} \frac{\cos x}{\sin^2 x} \, dx \text{ provided the limit is a finite number.}
\]

Otherwise, the integral diverges.

(c) If the improper integral in part (a) converges, then use your definition to calculate the value to which it converges. If the improper integral in part (a) does not converge, then explain why not. Show your work.

\[
\int_{a}^{\pi/3} \frac{\cos x}{\sin^2 x} \, dx = \int_{\sin(a)}^{\sin(\pi/3)} \frac{1}{u^2} \, du = -\frac{1}{u} \bigg|_{\sin(a)}^{\sin(\pi/3)} = -\frac{1}{\sin(\pi/3)} + \frac{1}{\sin(a)}
\]

by \( u = \sin(x) \), \( du = \cos(x) \, dx \).

\[
\lim_{a \to 0^+} \int_{a}^{\pi/3} \frac{\cos x}{\sin^2 x} \, dx = -\frac{1}{\sin(\pi/3)} + \lim_{a \to 0^+} \frac{1}{\sin(a)} \text{ and the last limit does not exist since } \sin(a) \to 0 \text{ as } a \to 0^+. \text{ Therefore, the integral diverges.}
\]