6. (20 points) The quantity $\int_1^\infty \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}}$ roughly models the resistance that football-shaped plankton encounter when falling through water. Note that $a=1$, $b=2$, and $c=3$ are constants that describe the dimensions of the plankton.

Find a value of M for which $\int_1^M \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}}$ differs from the original model of resistance by at most 0.001. *Hint*: make use of the integral $\int_M^\infty \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}}$ and the comparison test.

We want $\int_1^\infty \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}} - \int_1^M \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}}$

$$= \int_M^\infty \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}} < 0.001$$

We observe that $\int_M^\infty \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}} < \int_M^\infty \frac{dx}{\sqrt{(x)(x)(x)}} = \int_M^\infty \frac{1}{x^{\frac{3}{2}}}dx$ since a^2, b^2, and c^2 are all positive constants. So if $\int_M^\infty \frac{1}{x^{\frac{3}{2}}}dx < 0.001$ then $\int_M^\infty \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}} < 0.001$.

Convert to a proper integral and solve:

$$\lim_{b\to\infty}\int_M^b x^{-\frac{3}{2}}dx = \lim_{b\to\infty} -2x^{-\frac{1}{2}}\Big|_M^b = \lim_{b\to\infty}\left(\frac{-2}{\sqrt{b}}+\frac{2}{\sqrt{M}}\right) = \frac{2}{\sqrt{M}} < 0.001.$$

Algebra yields $M > \left(\frac{2}{0.001}\right)^2 = 4{,}000{,}000$