1. [11 points] There is a classic result in mathematics, which states that the number of prime numbers less than any number $x \geq 2$ is approximated by the function $\mathrm{li}(x) = \int_2^x \frac{dt}{\ln t}$.

a. [3 points] Is $\mathrm{li}(x)$ increasing, decreasing, or neither for $x \geq 2$? Provide justification for your answer.

Solution: This function is increasing. To see that it is increasing, we take its derivative

$$\frac{d}{dx}\mathrm{li}(x) = \frac{d}{dx}\int_2^x \frac{dt}{\ln t} = \frac{1}{\ln x} > 0.$$

-OR-
Since $\ln x$ is positive for $x \geq 2$, we know $f(x) = \frac{1}{\ln x} > 0$. Since $f(x)$ is the derivative of $\mathrm{li}(x)$, we know $\mathrm{li}(x)$ is increasing.

b. [3 points] Is $\mathrm{li}(x)$ concave up, concave down, or neither for $x \geq 2$? Provide justification for your answer.

Solution: This function is concave down. To see it is concave down, take the second derivative:

$$\frac{d^2}{dx^2}\mathrm{li}(x) = \frac{d}{dx}\frac{1}{\ln x} = -\frac{1}{x\ln^2 x} < 0.$$

-OR-
Since $\ln x$ is an increasing function, we know $f(x) = \frac{1}{\ln x}$ is decreasing, which means $f'(x) < 0$. Since $f'(x)$ is the second derivative of $\mathrm{li}(x)$, we know $\mathrm{li}(x)$ is concave down.

c. [5 points] Using Integration by Parts, put $\mathrm{li}(x)$ into the form

$$\mathrm{li}(x) = f(x) + \int_2^x \frac{dt}{(\ln t)^2}.$$

What is $f(x)$?

Solution: The way to get the second integral from the first looks like integration by parts. So we integrate $\int_2^x \frac{dt}{\ln t}$ by parts with $u = \frac{1}{\ln t}$ and $dv = dt$, giving $du = -\frac{dt}{t(\ln t)^2}$ and $v = t$. We get

$$t\frac{1}{\ln t}\Big|_2^x + \int_2^x \frac{dt}{\ln^2 t} = \frac{x}{\ln x} - \frac{2}{\ln 2} + \int_2^x \frac{dt}{\ln^2 t}.$$

So $f(x) = \frac{x}{\ln x} - \frac{2}{\ln 2}$.