2. [13 points] An ice cream cone has a height of 15 centimeters and the diameter of the top is 5 centimeters. The cone is filled with soft-serve ice cream such that the ice cream completely fills the cone, but does not exceed the top of the cone. The ice cream has a constant density of 2 grams per cubic centimeter.

a. [5 points] Write an expression for the approximate mass of ice cream contained in a circular cross-sectional slice that is located h_i centimeters from the the bottom tip of the cone and has depth Δh centimeters . Your answer may be in terms of h_i and Δh. Don't forget to include units.

Solution: Using similar triangles, at h_i centimeters from the tip of the cone, the radius is $\frac{1}{6}h_i$. This means the volume of a slice of depth Δh at height h_i is approximately $\frac{\pi}{36}h_i^2\Delta h$. Therefore, the mass of the slice is volume times density, which gives $\frac{2\pi}{36}h_i^2\Delta h = \frac{\pi}{18}h_i^2\Delta h$ grams.

b. [4 points] Set up a definite integral that can be used to determine the EXACT total mass of ice cream that is filling the cone, then solve for this exact value. Include appropriate units in your answer.

Solution: By summing up all such slices as found in part (a) and letting $\Delta h \to 0$, we have

$$\text{Total Mass} = \int_0^{15} \frac{\pi}{18}h^2 dh = \frac{\pi}{54}h^3|_0^{15} = \frac{3375\pi}{54} = \frac{125\pi}{2}\text{grams.}$$

c. [4 points] At what height above the tip of the cone is the center of mass of the ice cream? Give an EXACT answer, show all work, and include appropriate units.

Solution: We already found the total mass in part (b), but we need to calculate the total moments. We have

$$\int_0^{15} \frac{\pi}{18}h^3 dh = \frac{\pi}{72}h^4|_0^{15} = \frac{50625\pi}{72} = \frac{5625\pi}{8}.$$

Putting this together, we get that

$$\overline{h} = \frac{\frac{5625\pi}{8}}{\frac{125\pi}{2}} = \frac{45}{4} \text{ cm from the bottom of the cone.}$$