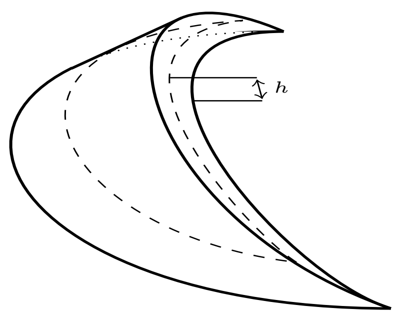
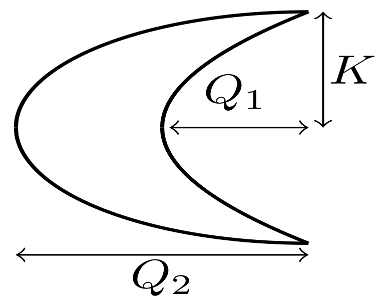
8. [12 points] Sand dunes come in many shapes. *Barchan* dunes, which have the shape shown on the left, are studied extensively by geomorphologists. Horizontal cross-sections of these dunes are crescent-shaped (the dashed line encloses one such cross-section), and can be approximated as the shape on the right. The area of this shape is given by the formula $A_h = K(\frac{\pi}{2}Q_2 - \frac{4}{3}Q_1)$.





You are studying a barchan dune of 10 meters height, for which the values of Q_1, Q_2, and K vary with respect to the height h (in meters) of the cross-section according to the functions $Q_1(h) = 10 - h$, $Q_2(h) = 20 - 2h$, $K(h) = 100 - h^2$. The density of sand in the dune is $\delta = 1600$ kilograms per cubic meter.

a. [5 points] Write an expression for the volume of one slice of sand dune h meters above the ground and Δh meters thick.

Solution:

$$V_{slice} \approx A_h \Delta h = (100 - h^2)\left[\frac{\pi}{2}(20 - 2h) - \frac{4}{3}(10 - h)\right]\Delta h.$$

b. [5 points] Write a definite integral that represents the total mass of sand in the dune. You do not need to evaluate this integral.

Solution: Height of the dune = 10, so

$$M_{dune} = \int_0^{10} 1600(100 - h^2)\left[\frac{\pi}{2}(20 - 2h) - \frac{4}{3}(10 - h)\right] dh.$$

c. [2 points] Write an expression (involving integrals) for the height of the center of mass of the sand dune. You do not need to evaluate this integral.

Solution:

$$\bar{h} = \frac{\int_0^{10} 1600h(100 - h^2)\left[\frac{\pi}{2}(20 - 2h) - \frac{4}{3}(10 - h)\right] dh}{\int_0^{10}(100 - h^2)\left[\frac{\pi}{2}(20 - 2h) - \frac{4}{3}(10 - h)\right] dh}$$