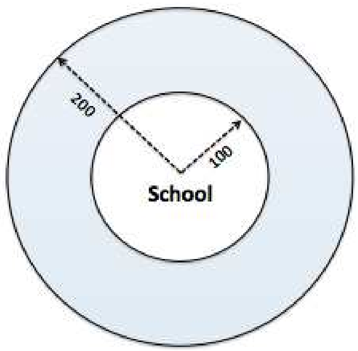
4. [8 points] In a small town, property values close to the school are determined primarily by how far the land is from the school. The function $\delta(r) = \dfrac{1}{ar^2+1}$ gives the value of the land (in thousands of dollars per m^2), where r is the distance (in meters) from the school and a is a positive contant.

a. [5 points] Find a formula containing a definite integral that computes the value of the land that lies in the annulus of inner radius 100 m and outer radius 200 m (figure shown below).



Solution: A thin annular slice has area $A_{slice} \approx 2\pi r\Delta r$, and so has an approximate value $V_{slice} \approx \frac{1}{ar^2+1}2\pi r\Delta r$. Summing these slice up and taking the limit as $\Delta r \to 0$, we get the integral

$$\text{Value of the land in the annulus} = \lim_{\Delta r\to 0}\sum \frac{1}{ar^2+1}2\pi r\Delta r = \int_{100}^{200} 2\pi r\frac{1}{ar^2+1}dr.$$

b. [3 points] Calculate the exact value of the land that lies in the annulus of inner radius 100 m and outer radius 200 m. Your answer should contain a. Show all your work.

Solution: Let $u = ar^2+1$. Then $du = 2ardr$. Our integral becomes

$$\int_{a100^2+1}^{a200^2+1} 2\pi r\frac{1}{u}\frac{du}{2ar} = \frac{\pi}{a}\int_{10{,}000a+1}^{40{,}000a+1}\frac{du}{u} = \frac{\pi}{a}\left(\ln(40,000a+1) - \ln(10,000a+1)\right).$$