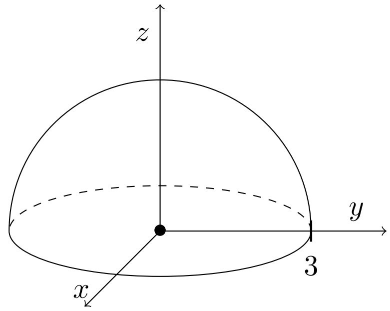
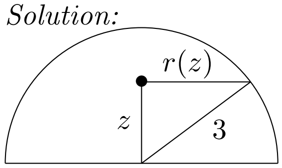
3. [15 points] Consider a hemisphere of radius 3m shown below. The hemisphere is filled to the top with water. The density of the water is 1000 kg/m^3.



a. [4 points] Find an expression for the mass of a circular slice of thickness Δz that is z meters above the base of the hemisphere.

Solution:



Using the Pythagorean formula $r(z) = \sqrt{9 - z^2}$. We have Mass $= 1000\pi r(z)^2 \Delta z$. Plugging in $r(z)$ we have Mass $= 1000\pi(9 - z^2)\Delta z$.

b. [7 points] What is the center of mass of the hemisphere of water? Justify your answers. Please limit any verbal explanation to a sentence or two.

Solution:
$\bar{x} = 0$ and $\bar{y} = 0$ because density is constant and the hemisphere is symmetric about the x and y axes.

$$\bar{z} = \frac{\int_0^3 1000\pi z(9 - z^2)dz}{\int_0^3 1000\pi(9 - z^2)dz} = \frac{\int_0^3 z(9 - z^2)dz}{\int_0^3 (9 - z^2)dz} = 9/8.$$

c. [4 points] Suppose water is evaporating from the hemisphere and the height of the water is decreasing at a constant rate of 1 m/day. Assuming $0 \le t < 3$, write an expression involving integrals which gives the z-coordinate of the center of mass of the water, t days after the water started evaporating. Do not evaluate any integrals.

Solution: The height of the water at time t is $3-t$. Thus we now integrate from 0 to $3-t$. $\bar{z} = \dfrac{\int_0^{3-t} 1000\pi z(9-z^2)dz}{\int_0^{3-t} 1000\pi(9-z^2)dz} = \dfrac{\int_0^{3-t} z(9-z^2)dz}{\int_0^{3-t}(9-z^2)dz}$